Couverture quadratique en marché incomplet pour des processus à accroissements indépendants et applications au marché de l'électricté.

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Variance Optimal Hedging in incomplete market for processes with independent increments and applications to electricity market.

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Titre: Couverture quadratique en marché incomplet pour des processus à accroissents indépendants et applications au marché de l'électricité.

Résumé: La thèse porte sur une décomposition explicite de Föllmer-Schweizer d'une classe importante d'actifs conditionnels lorsque le cours du sous-jacent est un processus à accroissements indépendants ou une exponentielle de tels processus. Ceci permet de mettre en oeuvre un algorithme efficace pour établir des stratégies optimales dans le cadre de la couverture quadratique. Ces résultats ont été implémentés dans le cas du marché de l'électricité.

Titolo: Copertura sulla base dello scarto quadratico medio nei mercati incompleti per dei processi a incrementi indipendenti e applicazioni al mercato elettrico.

Riassunto: In questa tesi di dottorato di ricerca vengono calcolate esplicitamente le scomposizioni dette di *Föllmer-Schweizer* per una famiglia significativa di opzioni finanziarie quando il prezzo del soggiacente é un processo a incrementi indipendenti o un esponenziale di tali processi. Le formule ottenute permettono di produrre un algoritmo efficiente per la risoluzione del problema della copertura che minimizza lo scarto quadratico medio nei mercati incompleti. I risultati sono stati implementati numericamente nell'ambito del mercato elettrico.

Title: Variance Optimal Hedging in incomplete market for processes with independent increments and applications to electricity market.

Abstract: For a large class of vanilla contingent claims, we establish an explicit Föllmer-Schweizer decomposition when the underlying is a process with independent increments (PII) and an exponential of a PII process. This allows to provide an efficient algorithm for solving the mean variance hedging problem. Applications to models derived from the electricity market are performed.

Key words and phrases: Variance-optimal hedging, Föllmer-Schweizer decomposition, Lévy process, Cumulative generating function, Characteristic function, Normal Inverse Gaussian process, Electricity markets, Incomplete Markets, Process with independent increments, trading dates optimization.

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Chapter 1

Introduction

Dans cette thèse, nous nous intéresserons aux problèmes de la couverture d'options en marché incomplet et à ses applications, notamment sur le marché Spot de l'électricité. En effet, la motivation première de cette thèse a été que sur le marché de l'électricité les pics de prix des actifs sont à la fois fréquents et élevés. Comme nous pouvons le voir dans la figure 1.1, la présence de sauts dans les prix de certains sous-jacents justifie l'utilisation de modèles non gaussiens et, entre autre, l'utilisation de processus à accroissements indépendants dans nos modèles de prix, afin de pouvoir représenter ces sauts. Il est clair que des variations de prix comme celles-là ne peuvent pas être expliquées par un modèle gaussien.

Du point de vue de la couverture, les modèles gaussiens correspondent en général aux marchés complets ou aux marchés qui peuvent être complétés. Or, l'utilisation de modèles non gaussiens utilisant, par exemple, des processus à accroissements indépendants entraine l'incomplétude du marché; c'est à dire un marché où les méthodes classiques de couverture et de valorisation du type de celle de Black et Scholes ne permettent plus une réplication parfaite des produits dérivés.

La question de la valorisation et de la couverture d'option en temps continu ou discret dans la cas non gaussien se pose donc. Quel est l'apport de la prise en compte des pics de prix du sous-jacent dans le calcul de couverture par rapport à la solution donnée par la formule de Black et Scholes? Comment se traduit, en terme d'erreur, le fait de discrétiser une stratégie de couverture optimale en temps continu?

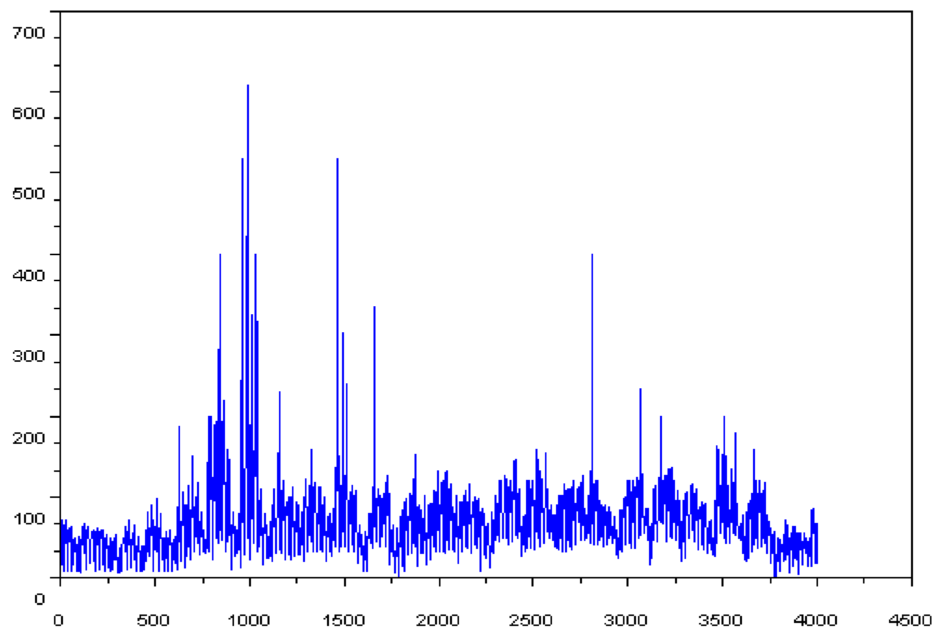


Figure 1.1: Prix du marché Spot de l'électricité sur le marché PowerNext entre le 15/11/05 et le 31/03/06 en euros par Mwh, heure par heure.

L'approche Variance-Optimale

Une approche populaire pour résoudre le problème de couverture en marché incomplet est celle de la couverture variance-optimale introduite dans [30]. Soit $(\Omega, \mathcal{F}, \mathbb{P})$ un espace de probabilité, soit $T > 0$ une maturité, posons S une $(\mathcal{F}_t)$-semimartingale de décomposition de Doob $S_t = S_0 + M_t + A_t$ pour tout $t \in [0, T]$. Appelons Θ l'espace des processus prévisibles $(v_t)_{t\in[0,T]}$ pour lequel l'intégrale stochastique $G_t(v) = \int_0^t v_s dS_s$ est une semimartingale de carré intégrable. Fixons une variable aléatoire de carré intégrable H. Le problème de couverture variance-optimale consiste à trouver une constante $c \in \mathbb{R}$ et une stratégie de couverture $(v_t)_{t\in[0,T]} \in \Theta$ qui minimisent le risque quadratique globale de couverture suivant:

$$\mathbb{E}[(H - c - G_T(v))^2]$$

En termes financiers, c répresente la valeur optimale du capital initial nécessaire pour minimiser notre erreur globale de couverture. φ^c représente la stratégie optimale d'achat et de vente sur le marché de l'actif H à chaque instant de couverture.

Richardson [30]; Schweizer [72, 73, 76, 66]; Gourieroux, Laurent et Pham [41] Cont, Tankov et Voltchkovaet [23] ou plus récemment Cerny et Kallsen [17] ont contribué de façon significative à la résolution de ce problème.

La décomposition de Föllmer-Schweizer est un outil classique utilisé pour résoudre le problème de couverture variance-optmale.

La décomposition de Föllmer-Schweizer

Définition. *On dit qu'une variable aléatoire $H \in \mathcal{L}^2(\Omega, \mathcal{F}, P)$ admet une **décomposition de Föllmer-Schweizer** si elle peut être représentée sous la forme suivante:*

$$H = H_0 + \int_0^T \xi_s^H dS_s + L_T^H \ , \quad P - a.s. \ , \tag{0.1}$$

où $H_0 \in \mathbb{R}$ est une constante, $\xi^H \in \Theta$ et $L^H = (L_t^H)_{t\in[0,T]}$ une martingale de carré intégrable telle que $\mathbb{E}[L_0^H] = 0$ et fortement orthogonale à la partie martingale locale M (i.e. $\langle L, M\rangle = 0$) apparaissant dans la décomposition de Doob de S.

Le premier article introduisant cette décomposition dans le cas où (S_t) est continue est celui de Föllmer-Schweizer [36]. Nous pouvons remarquer que dans le cas où (S_t) est une martingale de carré intégrable alors la décomposition de Föllmer-Schweizer coïncide avec la décomposition de Kunita-Watanabe.

L'existence d'une telle décomposition est primordiale dans la caractérisation de la solution de notre problème de couverture variance-optimale. En effet, c'est grâce au triplet (H_0, ξ, L) intervenant dans la décomposition de Föllmer-Schweizer que nous caractériserons la solution explicite de notre problème de couverture variance-optimale. On en déduit qu'une première étape nécessaire à la résolution de notre problème de couverture est de démontrer l'existence d'une telle décomposition pour notre semimartingale (S_t). Il convient tout d'abord de vérifier une condition introduite par Schweizer dans [72] appelée **condition de structure**.

Définition. *On dit que la semimartingale $(S_t)_{t\in[0,T]}$ satisfait la* **condition de structure** *(SC) s'il existe un processus prévisible $(\alpha_t)_{t\in[0,T]}$ tel que pour tout $t \in [0,T]$ on ait*

$$A_t = \int_0^t \alpha_s d\langle M\rangle_s, \quad K_T < \infty \ a.s.,$$

où l' on note

$$K_t = \int_0^t \alpha_s^2 d\langle M\rangle_s.$$

Dans ce cas, la $(\mathcal{F}_t)$-semimartingale (S_t) peut s'écrire sous la forme :

$$S_t = S_0 + M_t + \int_0^t \alpha_s d\left\langle M\right\rangle_s.$$

Le processus $(K_t)_{t\in[0,T]}$ joue un role important dans l'existence de la décomposition de Föllmer-Schweizer. Ce processus est appelé processus **mean-variance tradeoff**. Il est inspiré de la théorie en temps discret introduite dans [70] et définie en temps continu dans [36] puis [72]. Monat et Strciker, dans [61], ont donné une condition suffisante à l'existence et à l'unicité de la décomposition de Föllmer-Schweizer d'une variable aléatoire H.

Proposition. *Supposons que $(S_t)_{t\in[0,T]}$ satisfasse la* **condition de structure** *(SC) et que le processus mean-variance tradeoff K soit uniformement borné en t et ω alors toute variable aléatoire $H \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$ admet une unique décomposition de Föllmer-Schweizer.*

Ce résultat nous permet donc sous certaines conditions sur notre sous-jacent (S_t) de prouver l'existence de la décomposition de Föllmer-Schweizer de toute variable alétaoire H. L'existence de cette décomposition va nous permettre de prouver l'existence de la solution de notre problème de couverture variance-optimale.

La solution de notre problème de couverture variance-optimale

En effet, l'existence de la décomposition de Föllmer-Schweizer sous les conditions précédentes aboutit à l'existence de la solution de notre problème de couverture variance-optimale. Monat et Stricker, toujours dans [61], ont ainsi démontré le résultat suivant:

Théorème. *Supposons que* $(S_t)_{t\in[0,T]}$ *satisfasse la condition de structure (SC) et que le processus mean-variance tradeoff* K *soit uniformement borné en* t *et* ω *alors pour toute variable aléatoire* $H \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$*, il existe un unique couple* $(c^{(H)}, \varphi^{(H)}) \in \mathcal{L}^2(\mathcal{F}_0) \times \Theta$ *tel que*

$$\mathbb{E}[(H - c^{(H)} - G_T(\varphi^{(H)}))^2] = \min_{(c,v)\in\mathcal{L}^2(\mathcal{F}_0)\times\Theta} \mathbb{E}[(H - c - G_T(v))^2] \ .$$

Schweizer, dans [72], donne, dans le cas où le processus mean-variance tradeoff (K_t) est déterministe, une forme implicite (mais exploitable numériquement) du couple $(c^{(H)}, \varphi^{(H)}) \in \mathcal{L}^2(\mathcal{F}_0) \times \Theta$ solution du problème variance-optimale, ainsi que la valeur de la variance de notre erreur de couverture variance-optimale.

Théorème. *Supposons que* $(S_t)_{t\in[0,T]}$ *satisfasse la condition de structure (SC) et que le processus mean-variance tradeoff* (K_t) *soit déterministe. Soit* α *le processus prévisible apparaissant dans la condition de structure et* $H \in \mathcal{L}^2$ *la variable aléatoire admettant une décomposition de Föllmer-Schweizer; alors nous avons*

1. *Pour tout* $c \in \mathbb{R}$ *la stratégie optimale* $\varphi^{(c)} \in \Theta$ *solution de notre problème de couverture variance optimale est donnée par*

$$\varphi_t^{(c)} = \xi_t^H + \frac{\alpha_t}{1 + \Delta K_t}(H_{t-} - c - G_{t-}(\varphi^{(c)})) \ , \quad \textit{pour} \quad \textit{tout } t \in [0, T]$$

où le processus $(H_t)_{t\in[0,T]}$ *est défini par*

$$H_t := H_0 + \int_0^t \xi_s^H dX_s + L_t^H \ .$$

2. *De plus la variance de notre erreur de couverture variance-optimale vaut*

$$\min_{v\in\Theta} \mathbb{E}[(H - c - G_T(v))^2] = \mathcal{E}(-\tilde{K}_T)\left((H_0 - c)^2 + \mathbb{E}[(L_0^H)^2] + \int_0^T \frac{1}{\mathcal{E}(-\tilde{K}_s)} d\left(\mathbb{E}[\langle L^H \rangle_s]\right)\right) \tag{0.2}$$

où $\mathcal{E}(S)$ *est l'exponentielle de Doléans-Dade de la semimartingale* S *(voir séction II.8 p. 85 de [63]) et*

$$\tilde{K}_t = \int_0^t \frac{|\alpha_s|^2}{1 + \Delta K_s} d\langle M \rangle_s = \int_0^t \frac{1}{1 + \Delta K_s} dK_s, \text{ for all} t \in [0, T].$$

3. En particulier, si $\langle M, M\rangle$ est continue, nous avons alors que

$$\begin{aligned}\min_{v\in\Theta}\mathbb{E}[(H-c-G_T(v))^2] &= \exp(-K_T)\left((H_0-c)^2+\mathbb{E}[(L_0^H)^2]\right)\\ &\quad +\mathbb{E}\left[\int_0^T \exp\{-(K_T-K_s)\}d\left\langle L^H\right\rangle_s\right].\end{aligned}$$

Nous pouvons remarquer que dans le cas où la semimartingale (S_t) est continue, traitée dans [36], aucune condition sur K n'est requise. Plus récemment, une quantité importante de travaux traitant les problèmes de minimisation du risque local ou global ont été publiés. Il est donc impossible de tous les lister. Cependant nous pouvons citer [76], [9] et [17] qui comportent une bibliographie importante.

Une autre approche envisagée pour résoudre le problème de couverture variance optimale est celle de Cont, Tankov et Voltchkova dans [23], qui minimisent cette variance sous une mesure de probabilité équivalente par rapport à laquelle (S_t) est une martingale.

Le problème de couverture variance-optimale peut aussi être relié à la théorie des équations différentielles stochastiques rétrogrades (BSDEs) dans le sens de Pardoux et Peng [62], et a été proposé par Schweizer [72]. Dans [62], est considérée une équation différentielle stochastique rétrograde dirigée par un mouvement brownien. Dans [72], le mouvement brownien est remplacé par M. Le premier auteur ayant considéré une équation différentielle stochastique rétrograde dirigée par une martingale est Buckdahn dans [14].

Supposons $V_t = \int_0^t \alpha_s d\langle M\rangle_s$. Le problème de couverture variance-optimale consiste à trouver un triplet (V, ξ, L) résolvant la BSDE suivante

$$V_t = H - \int_t^T \xi_s dM_s - \int_t^T f(\omega, s, V_s, \xi_s)d\langle M\rangle_s - (L_T - L_t),$$

où

- $f(\omega, s, V_s, \xi_s) = \xi_s\alpha_s$
- $\mathbb{E}[V_t^2] < \infty$ pour tout $t \in [0, T]$
- $\mathbb{E}[\int_0^T \xi_s^2 d\langle M\rangle_s] < \infty$
- L est une $(\mathcal{F}_t)$-martingale locale orthogonale à M.

En fait, cette décomposition nous donne la solution au problème de minimisation du risque local de couverture [36]. Dans ce cas, V_t représente le prix de notre option à l'instant t et V_0 est l'espérance sous la *variance-optimal mesure* (VOM) de H.

La motivation du marché de l'électricité

Notre motivation à résoudre le problème de couverture variance optimale dans le cas de logarithme de prix à accroissements indépendants nous vient, comme nous l'avons mentionné précédement, du marché de l'électricité. En effet, du fait de l'impossibilité de stocker de l'électricité, l'un des instruments de couverture utilisé sont les prix à termes ou futur $F_t^{T_d}$ sur les prix spot (S_t). $F_t^{T_d}$ représente alors le prix futur à l'instant $t \leq T_d$ de la livraison de 1MWh d'électricité sur la période $[T_d, T_d + \theta]$.

Le modèle exponentiel de Lévy, proposé dans [11] et [21], permet de représenter à la fois la structure de volatilité et les pics de prix. Plus précisément, le prix futur est donné par le modèle à deux facteurs suivant:

$$F_t^{T_d} = F_0^{T_d} \exp(m_t^{T_d} + \int_0^t \underbrace{\sigma_S e^{-\lambda(T_d - s)} d\Lambda_s}_{facteur\ court\ terme} + \underbrace{\sigma_L W_t}_{facteur\ long\ terme}) \ , \quad \text{pour tout } t \in [0, T_d] \ , \quad (0.3)$$

où m est une tendance réelle déterministe, Λ un processus de Lévy réel et W un mouvement brownien réel. Nous remarquons que la dynamique des prix futurs $F_t^{T_d}$ est modélisée par une exponentielle de processus à accroissements indépendants.

Ceci justifie notre choix de nous intéresser à l'extension des résultats du problème de valorisation et de couverture variance-optimale dans le cas où le sous-jacent suit un modèle à accroissements indépendants mais non plus forcément stationnaires.

Cette thèse traitera donc du cas où le sous-jacent (S_t) est un processus à accroissements indépendants ou une exponentielle de processus à accroissements indépendants, et ceci dans le cas d'un marché en temps continu ou discret. Nous donnerons, entre autre, des formules explicites permettant d'obtenir le triplet (H_0, ξ, L) intervenant dans la décomposition de Föllmer-Schweizer et ceci dans le cas où la semimartingale (S_t) est une exponentielle de processus à accroissements indépendants et pour une classe particulère d'options introduite dans [49]. En effet, l'option H sera donnée par l'inverse d'une transformée de Laplace d'une fonction f contre une mesure complexe finie Π. Typiquement, on aura que $H = f(S_T)$ avec $f(s) = \int_{\mathbb{C}} s^z \Pi(dz)$. A titre d'exemple, nous avons qu'un call européen de strike K vérifie ce type de représentation et on a que pour $R > 1$ et $s > 0$

$$(s-K)^+ = \frac{1}{2i\Pi} \int_{R-i\infty}^{R+i\infty} s^z \frac{K^{1-z}}{z(z-1)} dz$$

Pour ce type d'option, nous exprimerons la valeur de notre stratégie de couverture variance-optimale $(\varphi_t^{(c)})_{t\in[0,T]}$ en fonction de la fonction cumulative génératrice $(\kappa_t)_{t\in[0,T]}$ du processus $X_t = log(S_t)$. (κ_t) étant définie pour l'ensemble des $z \in D := \{z \in \mathbb{C} \mid \mathbb{E}[e^{Re(z)X_t}] < \infty, \forall t \in [0,T]\}$ comme

$$\kappa_t : D \to \mathbb{C} \ , \quad \text{avec} \quad e^{\kappa_t(z)} = \mathbb{E}[S_t^z] = \mathbb{E}[e^{zX_t}] \ ,$$

On peut trouver dans [49], des résultats concernant le cas où le sous-jacent semimartingale (S_t) est une exponentielle de processus de Lévy (donc à accroissements indépendants et stationnaires).

Cette thèse se composera donc de deux parties. Chacune d'entre elles faisant objet d'une soumission à publication.

L'approche en temps continu

Le premier chapitre visera à résoudre le problème en temps continu. Dans la section 2.2, nous introduirons, dans un premier temps, les notions intervenant dans la résolution du problème de la couverture variance-optimale et nous définirons la décomposition de Föllmer-Schweizer. Deux cas de sous-jacent (S_t) seront alors étudiés:

- La section 2.3 portera sur l'étude du cas où le sous-jacent (S_t) est donné par un processus à accroissements indépendants (X_t). Dans ce cas précis, nous travaillerons sur une classe d'options du type transformée de Fourier de notre procéssus à accroissements indépendants:

  $$H = f(S_T) = f(X_T) \quad \text{with} \quad f(x) = \int_{\mathbb{R}} e^{iux} \mu(du) \ , \quad \text{pour tout } x \in \mathbb{R} \ ,$$

  pour une certaine mesure signée finie μ. Le théorème 2.3.34 établira alors des formules explicites permettant d'obtenir la décomposition de Föllmer-Schweizer d'une variable aléatoire H vérifiant ce type de représentation.

- La section 2.4 traitera ensuite du cas où le sous-jacent (S_t) est donné par une exponentielle de processus à accroissements indépendants $S_t = \exp X_t$. Nous donnerons des résultats faisant intervenir la fonction génératrice cumulative de (X_t). Nous établirons alors grâce à ces résultats le théorème 2.4.24 donnant les formules explicites de la décomposition de Föllmer-Schweizer d'une variable aléatoire H définie comme l'inverse d'une transformée de Laplace d'une fonction f contre une mesure complexe finie Π. $H = f(S_T)$ avec $f(s) = \int_{\mathbb{C}} s^z \Pi(dz)$.

Puis dans la section 2.5, nous donnerons, dans un premier temps, dans le thèorème 2.5.1 la solution explicite au problème de couverture variance-optimale dans le cas où (S_t) est donné par un processus à accroissements indépendants (X_t). Dans un second temps, le théorème 2.5.2 formulera la solution dans le cas où (S_t) est donné par une exponentielle de processus à accroissements indépendants $(\exp(X_t))$. Nous établirons ensuite le théorème 2.5.4 donnant la valeur explicite de la variance de l'erreur de couverture variance-optimale dans le cas où (S_t) est donné par une exponentielle de processus à accroissements indépendants $(\exp(X_t))$.

La section 2.6 portera sur l'application des résultats obtenus au cas particulier du marché de l'électricité. En effet, comme nous l'avons vu précédemment les prix futurs $(F_t^{T_d})$ sont donnés par une exponentielle de processus à accroissements indépendants (X_t) définie pour tout $t \in [0, T_d]$ par (0.3):

$$X_t = m_t + X_t^1 + X_t^2 = m_t^{T_d} + \int_0^t \sigma_s e^{-\lambda(T_d - s)} d\Lambda_s + \sigma_l W_t \ .$$

Nous établirons ainsi les formules explicites de notre solution de couverture variance optimale au cas particulier du marché de l'électricité.

Enfin, la section 2.7 présentera des simulations numériques qui permettront d'illustrer et d'interpréter nos résultats.

L'approche en temps discret

Le second chapitre cherchera à résoudre le problème de la minimisation de la couverture variance-optimale en temps discret. Nous introduirons dans la section 3.2 les notions intervenant dans la résolution du problème de couverture variance-optimale. La version discrète de la décomposition de Föllmer-Schweizer ainsi que les conditions suffisantes à son existence seront ainsi définies.

Nous utiliserons ensuite dans la section 3.3 la version discrète de la fonction génératrice cumulative du processus $(S_n)_{n=0,1,\dots,N}$ pour établir la proposition 3.3.19 donnant la formule explicite de la décomposition discrete de Föllmer-Schweizer dans le cas d'un modèle exponentiel de processus à accroissements indépendants.

Dans un troisième temps, dans la section 3.4, nous établirons dans le théorème 3.4.1 la solution explicite du problème de couverture variance-optimale. Le théorème 3.4.3 permettra d'établir la formule explicite donnant la valeur de la variance de l'erreur de couverture variance-optimale.

Pour finir, la section 3.5 présentera des simulations numériques qui seront articulées en deux temps.

- Nous nous intéresserons tout d'abord au cas d'un payoff irrégulier (option digitale) avec un sous-jacent suivant un modèle exponentiel de processus à accroissements indépendants et stationnaires. Nous montrerons que le choix d'instants de couverture "équirépartis" sur $[0, T]$ n'est pas forcément optimal au vue du caractère irrégulier du payoff.

- Puis, dans une seconde partie, nous travaillerons dans le cas d'un payoff plus régulier (option call européen) mais avec un sous-jacent suivant un modèle exponentiel de processus à accroissements indépendants et non plus stationnaires. Le fait d'avoir une volatilité qui augmente en se rapprochant de la maturité T de l'option nous permettra de montrer que l'erreur de couverture variance-optimale peut être réduite en se couvrant plus souvent quand nous nous rapprocherons de T.

Une des conclusions des ces simulations sera que dans les deux cas nous arrivons à réduire l'erreur de couverture variance-optimale en optimisant nos instants de couverture.

Chapter 2

Variance-Optimal hedging in continuous time

This chapter is the object of the paper [45].

Abstract. *For a large class of vanilla contingent claims, we establish an explicit Föllmer-Schweizer decomposition when the underlying is a process with independent increments (PII) and an exponential of a PII process. This allows to provide an efficient algorithm for solving the mean variance hedging problem. Applications to models derived from the electricity market are performed.*

Key words and phrases: Variance-optimal hedging, Föllmer-Schweizer decomposition, Lévy processes, Cumulative generating function, Characteristic function, Normal Inverse Gaussian process, Electricity markets, Process with independent increments.

2000 AMS-classification: 60G51, 60H05, 60J25, 60J75

JEL-classification: C02, G11, G12, G13

2.1 Introduction

There are basically two main approaches to define the *mark to market* of a contingent claim: one relying on the *no-arbitrage assumption* and the other related to a *hedging portfolio*, those two approaches converging in the specific case of complete markets. A simple introduction to the different hedging and pricing models in incomplete markets can be found in chapter 10 of [22].

The fundamental theorem of Asset Pricing [26] implies that a pricing rule without arbitrage that moreover satisfies some usual conditions (linearity, non anticipativity ...) can always be written as an expectation under a martingale measure. In general, the resulting price is not linked with a hedging strategy except in some specific cases such as complete markets. More precisely, it is proved [26] that the market completeness is equivalent to uniqueness of the equivalent martingale measure. Hence, when the market is not complete, there exist several equivalent martingale measures (possibly an infinity) and one has to specify a criterion to select one specific pricing measure: to recover some given option prices (by calibration) [44]; to simplify calculus and obtain a simple process under the pricing measure; to maintain the structure of the real world dynamics; to minimize a *distance* to the objective probability (entropy [38] ...). In this framework, the difficulty is to understand in a practical way the impact of the choice of the martingale measure on the resulting prices.

If the resulting price is in this case not directly connected to a hedging strategy, yet it is possible to consider the hedging question in a second step, optimizing the hedging strategy for the given price. In this framework, one approach consists in deriving the hedging strategy minimizing the *global quadratic hedging error* under the pricing measure where the martingale property of the underlying highly simplifies calculations. This approach, is developed in [22], in the case of exponential-Lévy models: the optimal quadratic hedge is then expressed as a solution of an integro-differential equation involving the Lévy measure. Unfortunately, minimizing the quadratic hedging error under the pricing measure has no clear interpretation since the resulting hedging strategy can lead to huge quadratic error under the objective measure. On the other hand [23] continues this approach, again in the martingale framework, providing some interesting financial motivations.

Alternatively, one can define option prices as a by-product of the hedging strategy. In the case of complete markets, any option can be replicated perfectly by a self-financed hedging portfolio continuously rebalanced, then the option *hedging value* can be defined as the cost of the hedging strategy. When the market is not complete, it is not possible, in general, to hedge perfectly an option. One has to specify risk criteria, and consider the hedging strategy that minimizes the distance (in terms of the given criteria) between the pay-off of the option and the terminal value of the hedging portfolio. Then, the price of the option is related to the cost of this imperfect hedging strategy to which is added in practice another prime related to the residual risk induced by incompleteness.

Several criteria can be adopted. The aim of super-hedging is to hedge all cases. This approach yields in general prices that are too expensive to be realistic [32]. Quantile hedging modifies this approach allowing for a limited probability of loss [34]. Indifference utility pricing introduced in [47] defines the price of an option to sell (resp. to buy) as the minimum initial value s.t. the hedging portfolio with the option sold (resp. bought) is equivalent (in term of utility) to the initial portfolio. Quadratic hedging is developed in [72], [74]: the quadratic distance between the hedging portfolio and the pay-off is minimized. Then, contrarily to the case of utility maximization, losses and gains are treated in a symmetric manner, which yields a *fair price* for both the buyer and the seller of the option.

In this paper, we follow this last approach and our developments can be used in both the *no-arbitrage value* and the *hedging value* framework: either to derive the hedging strategy minimizing the *global quadratic hedging error* under the objective measure, for a given pricing rule; or to derive both the price and the hedging strategy minimizing the *global quadratic hedging error* under the objective measure.

We spend now some words related to the global quadratic hedging approach which is also called *mean-variance hedging* or *global risk minimization*. Given a square integrable r.v. H, we say that the pair (V_0, φ) is optimal if $(c, v) = (V_0, \varphi)$ minimizes the functional $\mathbb{E}\left(H - c - \int_0^T v dS\right)^2$. The price V_0 represents the price of the contingent claim H and φ is the optimal strategy.

Technically speaking, the global risk minimization problem, is based on the so-called *Föllmer-Schweizer* decomposition (or FS decomposition) of a square integrable random variable (representing the contingent claim) with respect to an $(\mathcal{F}_t)$-semimartingale $S = M + A$ modeling the asset price: M is an $(\mathcal{F}_t)$-local martingale and A is a bounded variation process with $A_0 = 0$. Mathematically, the FS decomposition, constitutes the generalisation of the martingale representation theorem (Kunita-Watanabe representation) when S is a Brownian motion or a martingale. Given a square integrable random variable H, the problem consists in expressing H as $H_0 + \int_0^T \xi dS + L_T$ where ξ is predictable and L_T is the terminal value of an orthogonal martingale L to M, i.e. the martingale part of S. The seminal paper is [36] where the problem is treated in the case that S is continuous. In the general case S is said to have the **structure condition** (SC) condition if there is a predictable process α such that $A_t = \int_0^t \alpha_s d\langle M\rangle_s$ and $\int_0^T \alpha_s^2 d\langle M\rangle_s < \infty$ a.s. In the sequel most of contributions were produced in the multidimensional case. Here for simplicity we will formulate all the results in the one-dimensional case.

An interesting connection with the theory of backward stochastic differential equations (BSDEs) in the sense of [62], was proposed in [72]. [62] considered BSDEs driven by Brownian motion; in [72] the Brownian motion is in fact replaced by M. The first author who considered a BSDE driven by a martingale was [14]. Suppose that $V_t = \int_0^t \alpha_s d\langle M\rangle_s$. The BSDE problem consists in finding a triple (V, ξ, L) where

$$V_t = H - \int_t^T \xi_s dM_s - \int_t^T \xi_s \alpha_s d\langle M\rangle_s - (L_T - L_t),$$

and L is an $(\mathcal{F}_t)$-local martingale orthogonal to M.

In fact, this decomposition provides the solution to the so called local risk minimization problem, see [36]. In this case, V_t represents the *price* of the contingent claim at time t and the price V_0 constitutes in fact the expectation under the so called *variance optimal signed measure* (VOM). Hence, in full generality, the price V_0 is not guaranteed to be arbitrage-free. In case of continuous processes, the variance optimal measure is proved to be nonegative under a mild no-arbitrage condition [75]. Arai [3] and [2] provides sufficient conditions for the variance-optimal martingale measure to be a probability measure, for discontinuous semimartingales.

In the framework of FS decomposition, a process which plays a significant role is the so-called **mean variance tradeoff** (MVT) process K. This notion is inspired by the theory in discrete time started by [70]; in the continuous time case K is defined as $K_t = \int_0^t \alpha_s^2 d\langle M\rangle_s,\ t \in [0, T]$. [72] shows the existence of the mean-variance hedging problem if the MVT process is deterministic. In fact, a slight more general condition was the (ESC) condition and the EMVT process but we will not discuss here further details. We remark that in the continuous case, treated by [36], no need of any condition on K is required. When the MVT process is deterministic, [72] is able to solve the global quadratic variation problem and provides an efficient relation, see Theorem 2.5.2 with the FS decomposition. He also shows that, for the obtention of the mentioned relation, previous condition is not far from being optimal. The next important step was done in [61] where under the only condition that K is uniformly bounded, the FS decomposition of any square integrable random variable admits existence

and uniqueness and the global minimization problem admits a solution.
More recently has appeared an incredible amount of papers in the framework of global (resp. local) risk minimization, so that it is impossible to list all of them and it is beyond our scope. Two significant papers containing a good list of references are [76], [9] and [17].

For the sake of financial applications, one would like to find an expression for the FS decomposition as *explicit* as possible. We are not interested in generalizing the conditions under which the FS decomposition exists. Besides, the numerical computation of BSDE (and therefore of FS decomposition) is a real issue in applied probability and mathematical finance. We recall that Clark-Ocone formula provides an explicit form for the Kunita Watanabe decomposition (in the Brownian case). The present paper aims, in the spirit of a simplified Clark-Ocone formula, at providing an explicit form for the FS decomposition for a large class of European payoffs H, when the process S is a process with independent increments (PII) or an exponential of PII. In the case of Lévy processes, there are some Clark-Ocone type formula, but they are in a different spirit than ours. We acknowledge for instance [29, 58].

From a practical point of view, this serves to compute efficiently the variance optimal hedging strategy which is directly related to the FS decomposition, since the mean-variance tradeoff is for that type of processes deterministic. One major idea proposed by [49] in the case where the log price is a Lévy process consists in expressing the payoff as a linear combination of exponential payoffs for which the variance optimal hedging strategy can be expressed explicitly. We propose here to use the same idea of using Laplace transforms representation of payoff but to extend the results of [49] to the case of PII and exponential of PII price processes.

The first part of this paper puts emphasis on PII and contingent claims that are provided by some Fourier transform of a finite measure: an original approach is developed to derive explicit FS decompositions. The second part of this paper extends results of [49] concerning exponential of Lévy processes and contingent claims that are Laplace-Fourier transform of a finite measure to the case of exponential of PII. Restricting assumptions was a leading issue for this work. In particular, our results do not require any assumption on the absolute continuity of the cumulant generating function of $\log(S_t)$.
One practical motivation for considering processes with independent and possibly non stationary increments came from hedging problems in the electricity market. Because of non-storability of electricity, the hedging instrument is in that case, a forward contract with value $S_t^0 = e^{-r(T_d-t)}(F_t^{T_d} - F_0^{T_d})$ where $F_t^{T_d}$ is the forward price given at time $t \leq T_d$ for

delivery of 1MWh at time T_d. Hence, the dynamic of the underlying S^0 is directly related to the dynamic of forward prices. Now, forward prices are known to exhibit both heavy tails (especially on the short term) and a volatility term structure which require the use of models with both non Gaussian and non stationary increments.
The paper is organized as follows. After this introduction and some generalities about semimartingales, we introduce the notion of FS decomposition and describe local and global risk minimization. Then, we examine at Section 3 (resp. 4) the explicit FS decomposition for PII processes (resp. exponential of PII processes). Section 5 is devoted to the solution to the global minimization problem and Section 6 to the case of a model intervening in the electricity market. Section 7 is devoted to simulations.

2.2 Generalities on semimartingales and Föllmer-Schweizer decomposition

In the whole paper, $T > 0$, will be a fixed terminal time and we will denote by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ a filtered probability space, fulfilling the usual conditions.

2.2.1 Generating functions

Let $X = (X_t)_{t\in[0,T]}$ be a real valued stochastic process.

Definition 2.2.1. *The* **characteristic function** *of (the law of) X_t is the continuous mapping*

$$\varphi_{X_t} : \mathbb{R} \to \mathbb{C} \quad \textit{with} \quad \varphi_{X_t}(u) = \mathbb{E}[e^{iuX_t}] \ .$$

In the sequel, when there will be no ambiguity on the underlying process X, we will use the shortened notation φ_t for φ_{X_t}.

Definition 2.2.2. *The* **cumulant generating function** *of (the law of) X_t is the mapping $z \mapsto \mathrm{Log}(\mathbb{E}[e^{zX_t}])$ where $\mathrm{Log}(w) = \log(|w|) + i\mathrm{Arg(w)}$ where $\mathrm{Arg(w)}$ is the Argument of w, chosen in $]-\pi, \pi]$; Log is the principal value logarithm. In particular we have*

$$\kappa_{X_t} : D \to \mathbb{C} \quad \textit{with} \quad e^{\kappa_{X_t}(z)} = \mathbb{E}[e^{zX_t}] \ ,$$

where $D := \{z \in \mathbb{C} \mid \mathbb{E}[e^{Re(z)X_t}] < \infty, \ \forall t \in [0,T]\}$.

In the sequel, when there will be no ambiguity on the underlying process X, we will use the shortened notation κ_t for κ_{X_t}.

We observe that D includes the imaginary axis.

Remark 2.2.3. *1. For all $z \in D$, $\kappa_t(\bar{z}) = \overline{\kappa_t(z)}$, where $\bar{z}$ denotes the conjugate complex of $z \in \mathbb{C}$.*

2. For all $z \in D \cap \mathbb{R}$, $\kappa_t(z) \in \mathbb{R}$.

2.2.2 Semimartingales

An $(\mathcal{F}_t)$-semimartingale $X = (X_t)_{t\in[0,T]}$ is a process of the form $X = M + A$, where M is an $(\mathcal{F}_t)$-local martingale and A is a bounded variation adapted process vanishing at zero. $||A||_T$ will denote the total variation of A on $[0, T]$. Given two $(\mathcal{F}_t)$- local martingales M and N, $\langle M, N\rangle$ will denote the angle bracket of M and N, i.e. the unique bounded variation predictable process vanishing at zero such that $MN - \langle M, N\rangle$ is an $(\mathcal{F}_t)$-local martingale. If X and Y are $(\mathcal{F}_t)$-semimartingales, $[X, Y]$ denotes the square bracket of X and Y, i.e. the quadratic covariation of X and Y. In the sequel, if there is no confusion about the underlying filtration $(\mathcal{F}_t)$, we will simply speak about semimartingales, local martingales, martingales. All the local martingales admit a cadlag version. By default, when we speak about local martingales we always refer to their cadlag version.
More details about previous notions are given in chapter I.1. of [53].

Remark 2.2.4. *1. All along this paper we will consider $\mathbb{C}$-valued martingales (resp. local martingales, semimartingales). Given two $\mathbb{C}$-valued local martingales M^1, M^2 then $\overline{M^1}, \overline{M^2}$ are still local martingales. Moreover $\langle \overline{M^1}, \overline{M^2}\rangle = \overline{\langle M^1, M^2\rangle}$.*

2. If M is a $\mathbb{C}$-valued martingale then $\langle M, \overline{M}\rangle$ is a real valued increasing process.

Theorem 2.2.5. *$(X_t)_{t\in[0,T]}$ is a real semimartingale iff the characteristic function, $t \mapsto \varphi_t(u)$, has bounded variation over all finite intervals, for all $u \in \mathbb{R}$.*

Definition 2.2.6. *An $(\mathcal{F}_t)$-**special semimartingale** is an $(\mathcal{F}_t)$-semimartingale X with the decomposition $X = M + A$, where M is a local martingale and A is a bounded variation predictable process starting at zero.*

Remark 2.2.7. *The decomposition of a special semimartingale of the form $X = M + A$ is unique, see [53] definition 4.22.*

For any special semimartingale X we define

$$||X||^2_{\delta^2} = \mathbb{E}\left[[M,M]_T\right] + \mathbb{E}\left(||A||^2_T\right) \ .$$

The set δ^2 is the set of $(\mathcal{F}_t)$-special semimartingale X for which $||X||^2_{\delta^2}$ is finite.

A **truncation function** defined on $\mathbb{R}$ is a bounded function $h : \mathbb{R} \to \mathbb{R}$ with compact support such that $h(x) = x$ in a neighbourhood of 0.

An important notion, in the theory of semimartingales, is the notion of characteristics, defined in definition II.2.6 of [53]. Let $X = M + A$ be a real-valued semimartingale. A **characteristic** is a triplet, (b, c, ν), depending on a fixed truncation function, where

1. b is a predictable process with bounded variation;
2. $c = \langle M^c, M^c \rangle$, M^c being the continuous part of M according to Theorem I.4.18 of [53].
3. ν is a predictable random measure on $\mathbb{R}^+ \times \mathbb{R}$, namely the compensator of the random measure μ^X associated to the jumps of X.

Given a real cadlag stochastic process X, the quantity ΔX_t will represent the jump $X_t - X_{t-}$.

2.2.3 Föllmer-Schweizer Structure Condition

Let $X = (X_t)_{t\in[0,T]}$ be a real-valued special semimartingale with canonical decomposition,

$$X = M + A \ .$$

For the clarity of the reader, we formulate in dimension one, the concepts appearing in the literature, see e.g. [72] in the multidimensional case.

Definition 2.2.8. *For a given local martingale M, the space $L^2(M)$ consists of all predictable $\mathbb{R}$-valued processes $v = (v_t)_{t\in[0,T]}$ such that*

$$\mathbb{E}\left[\int_0^T |v_s|^2 d\left\langle M\right\rangle_s\right] < \infty \ .$$

For a given predictable bounded variation process A, the space $L^2(A)$ consists of all predictable $\mathbb{R}$-valued processes $v = (v_t)_{t\in[0,T]}$ such that

$$\mathbb{E}\left[(\int_0^T |v_s| d||A||_s)^2\right] < \infty \ .$$

Finally, we set

$$\Theta := L^2(M) \cap L^2(A) \ .$$

For any $v \in \Theta$, the stochastic integral process

$$G_t(v) := \int_0^t v_s dX_s, \quad \text{for all } t \in [0,T] \ ,$$

is therefore well-defined and is a semimartingale in δ^2 with canonical decomposition

$$G_t(v) = \int_0^t v_s dM_s + \int_0^t v_s dA_s \ , \quad \text{for all } t \in [0,T] \ .$$

We can view this stochastic integral process as the gain process associated with strategy v on the underlying process X.

Definition 2.2.9. *The* ***minimization problem*** *we aim to study is the following. Given $H \in \mathcal{L}^2$, an admissible strategy pair (V_0, φ) will be called* ***optimal*** *if $(c,v) = (V_0, \varphi)$ minimizes the expected squared hedging error*

$$\mathbb{E}[(H - c - G_T(v))^2] \ , \tag{2.1}$$

over all admisible strategy pairs $(c,v) \in \mathbb{R} \times \Theta$. V_0 will represent the **price** *of the contingent claim H at time zero.*

Definition 2.2.10. *Let $X = (X_t)_{t\in[0,T]}$ be a real-valued special semimartingale. X is said to satisfy the* ***structure condition (SC)*** *if there is a predictable $\mathbb{R}$-valued process $\alpha = (\alpha_t)_{t\in[0,T]}$ such that the following properties are verified.*

1. $A_t = \int_0^t \alpha_s d\langle M\rangle_s \ , \quad$ *for all $t \in [0,T]$, so that $dA \ll d\langle M\rangle$.*
2. $\displaystyle\int_0^T \alpha_s^2 d\langle M\rangle_s < \infty \ , \quad P-a.s.$

Definition 2.2.11. *From now on, we will denote by $K = (K_t)_{t\in[0,T]}$ the cadlag process*

$$K_t = \int_0^t \alpha_s^2 d\langle M\rangle_s \ , \quad \textit{for all } t \in [0,T] \ .$$

This process will be called the ***mean-variance tradeoff*** *(MVT) process.*

Remark 2.2.12. *In [72], the process $(K_t)_{t\in[0,T]}$ is denoted by $(\widehat{K}_t)_{t\in[0,T]}$.*

Lemma 2 of [72] states the following.

Proposition 2.2.13. *If X satisfies (SC) such that $\mathbb{E}[K_T] < \infty$, then $\Theta = L^2(M)$.*

The structure condition (SC) appears quite naturally in applications to financial mathematics. In fact, it is mildly related to the no arbitrage condition. In fact (SC) is a natural extension of the existence of an equivalent martingale measure from the case where X is continuous. Next proposition will show that every adapted continuous process X admitting an equivalent martingale measure satisfies (SC).

Proposition 2.2.14. *Let X be a $(P, \mathcal{F}_t)$ continuous semimartingale. Suppose the existence of a locally equivalent probability $Q \sim P$ under which X is an $(Q, \mathcal{F}_t)$-local martingale, then (SC) is verified.*

Proof. Let $(D_t)_{t\in[0,T]}$ be the strictly positive continuous Q-local martingale such that $dP = D_T dQ$. By Theorem VIII.1.7 of [65], $M = X - \langle X, L\rangle$ is a continuous P-local martingale, where L is the continuous Q-local martingale associated to the density process i.e.

$$D_t = \exp\{L_t - \frac{1}{2}\langle L\rangle_t\} \ , \quad \text{for all } t \in [0,T] \ .$$

According to Lemma IV.4.2 in [65], there is a progressively measurable process R such that for all $t \in [0,T]$,

$$L_t = \int_0^t R_s dX_s + O_t \quad \text{and} \quad \int_0^T R_s^2 d\left\langle X\right\rangle_s < \infty \ , \quad Q-\text{a.s.} \ ,$$

where O is a Q-local martingale such that $\langle X, O\rangle = 0$. Hence,

$$\langle X, L\rangle_t = \int_0^t R_s d\langle X\rangle_s \quad \text{and} \quad X_t = M_t + \int_0^t R_s d[X]_s \ , \quad \text{for all } t \in [0,T].$$

We end the proof by setting $\alpha_t = \dfrac{d\langle X, L\rangle_t}{d\langle X\rangle_t} = R_t$. □

2.2.4 Föllmer-Schweizer Decomposition and variance optimal hedging

Throughout this section, as in Section 2.2.3, X is supposed to be an $(\mathcal{F}_t)$-special semimartingale fulfilling the (SC) condition.

We recall here the definition stated in Chapter IV.3 p. 179 of [63].

Definition 2.2.15. *Two $(\mathcal{F}_t)$-martingales M, N are said to be* ***strongly orthogonal*** *if MN is a uniformly integrable martingale.*

Remark 2.2.16. *If M, N are strongly orthogonal, then they are (weakly) orthogonal in the sence that $\mathbb{E}[M_T N_T] = 0$.*

Lemma 2.2.17. *Let M, N be two square integrable martingales. Then M and N are strongly orthogonal if and only if $\langle M, N\rangle = 0$.*

Proof. Let $\mathcal{S}(M)$ be the stable subspace generated by M. $\mathcal{S}(M)$ includes the space of martingales of the form

$$M_t^f := \int_0^t f(s) dM_s \ , \quad \text{for all } t \in [0, T] \ ,$$

where $f \in L^2(dM)$ is deterministic. According to Lemma IV.3.2 of [63], it is enough to show that, for any $f \in L^2(dM)$, $g \in L^2(dN)$, M^f and N^g are weakly orthogonal in the sense that $\mathbb{E}[M_T^f N_T^g] = 0$. This is clear since previous expectation equals

$$\mathbb{E}[\left\langle M^f, N^g\right\rangle_T] = \mathbb{E}\left(\int_0^T fgd\left\langle M, N\right\rangle\right) = 0$$

if $\langle M, N\rangle = 0$. This shows the converse implication.
The direct implication follows from the fact that MN is a martingale, the definition of the angle bracket and uniqueness of special semimartingale decomposition. □

Definition 2.2.18. *We say that a random variable $H \in \mathcal{L}^2(\Omega, \mathcal{F}, P)$ admits a* ***Föllmer-Schweizer (FS) decomposition****, if it can be written as*

$$H = H_0 + \int_0^T \xi_s^H dX_s + L_T^H \ , \quad P - a.s. \ , \tag{2.2}$$

where $H_0 \in \mathbb{R}$ is a constant, $\xi^H \in \Theta$ and $L^H = (L_t^H)_{t\in[0,T]}$ is a square integrable martingale, with $\mathbb{E}[L_0^H] = 0$ and strongly orthogonal to M.

We formulate for this section one basic assumption.

Assumption 1. *We assume that X satisfies (SC) and that the MVT process K is uniformly bounded in t and ω.*

The first result below gives the existence and the uniqueness of the Föllmer-Schweizer decomposition for a random variable $H \in \mathcal{L}^2(\Omega, \mathcal{F}, P)$. The second affirms that subspaces $G_T(\Theta)$ and $\{\mathcal{L}^2(\mathcal{F}_0) + G_T(\Theta)\}$ are closed subspaces of $\mathcal{L}^2$. The last one provides existence and uniqueness of the solution of the minimization problem (2.1). We recall Theorem 3.4 of [61].

Theorem 2.2.19. *Under Assumption 1, every random variable $H \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$ admits a FS decomposition. Moreover, $H_0 \in \mathbb{R}$, $\xi \in L^2(M)$ and L_T is uniquely determined by H.*

We recall Theorem 4.1 of [61].

Theorem 2.2.20. *Under Assumption 1, the subspaces $G_T(\Theta)$ and $\{\mathcal{L}^2(\mathcal{F}_0) + G_T(\Theta)\}$ are closed subspaces of $\mathcal{L}^2$.*

So we can project any random variable $H \in \mathcal{L}^2$ on $G_T(\Theta)$. By Theorem 2.2.19, we have the uniqueness of the solution of the minimization problem (2.1). This is given by Theorem 4.6 of [61], which is stated below.

Theorem 2.2.21. *We suppose Assumption 1.*

1. *For every $H \in \mathcal{L}^2(\Omega, \mathcal{F}, P)$ and every $c \in \mathcal{L}^2(\mathcal{F}_0)$, there exists a unique strategy $\varphi^{(c)} \in \Theta$ such that*

$$\mathbb{E}[(H - c - G_T(\varphi^{(c)}))^2] = \min_{v \in \Theta} \mathbb{E}[(H - c - G_T(v))^2] \ . \tag{2.3}$$

2. *For every $H \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$ there exists a unique $(c^{(H)}, \varphi^{(H)}) \in \mathcal{L}^2(\mathcal{F}_0) \times \Theta$ such that*

$$\mathbb{E}[(H - c^{(H)} - G_T(\varphi^{(H)}))^2] = \min_{(c,v) \in \mathcal{L}^2(\mathcal{F}_0) \times \Theta} \mathbb{E}[(H - c - G_T(v))^2] \ .$$

From Föllmer-Schweizer decomposition follows the solution to the global minimization problem (2.1). Next theorem gives the explicit form of the optimal strategy.

Theorem 2.2.22. *Suppose that X satisifies (SC) and that the MVT process K of X is deterministic and let α be the process appearing in Definition 2.2.10 of (SC). Let $H \in \mathcal{L}^2$ with FS-decomposition (2.2).*

1. *For any $c \in \mathbb{R}$, the solution of the minimization problem (2.3) verifies $\varphi^{(c)} \in \Theta$ for any $c \in \mathbb{R}$, such that*

$$\varphi_t^{(c)} = \xi_t^H + \frac{\alpha_t}{1 + \Delta K_t}(H_{t-} - c - G_{t-}(\varphi^{(c)})) \ , \quad \text{for all } t \in [0, T] \tag{2.4}$$

where the process $(H_t)_{t \in [0,T]}$ is defined by

$$H_t := H_0 + \int_0^t \xi_s^H dX_s + L_t^H \ . \tag{2.5}$$

2.

$$\min_{v\in\Theta}\mathbb{E}[(H-c-G_T(v))^2]=\mathcal{E}(-\tilde{K}_T)\left((H_0-c)^2+\mathbb{E}[(L_0^H)^2]+\int_0^T\frac{1}{\mathcal{E}(-\tilde{K}_s)}d\left(\mathbb{E}[\langle L^H\rangle_s]\right)\right)\tag{2.6}$$

where, given a semimartingale X, $\mathcal{E}(X)$ is the Doléans-Dade exponential of X, see section II.8 p. 85 of [63] and

$$\tilde{K}_t=\int_0^t\frac{|\alpha_s|^2}{1+\Delta K_s}d\langle M\rangle_s=\int_0^t\frac{1}{1+\Delta K_s}dK_s,\ \text{for all } t\in[0,T].$$

3. *In particular, if $\langle M,M\rangle$ is continuous,*

$$\begin{aligned}\min_{v\in\Theta}\mathbb{E}[(H-c-G_T(v))^2] &= \exp(-K_T)\left((H_0-c)^2+\mathbb{E}[(L_0^H)^2]\right)\\ &\quad+\mathbb{E}\left[\int_0^T\exp\{-(K_T-K_s)\}d\langle L^H\rangle_s\right].\end{aligned}$$

Remark 2.2.23. 1. *Point 1. is a consequence of Theorem 3 of [72] which in fact is stated under a more general condition, i.e. the so called (ESC) condition, which is associated with the extended mean-variance tradeoff (EVT) process $\tilde{K}$.*

2. *Point 2. is stated again under condition (ESC) in Corollary 9 of [72].*

3. *Since $\langle M,M\rangle$ is continuous, $\tilde{K}=K$ and $\mathcal{E}(K)=\exp(K)$ because K has bounded variation. This finally shows point 3.*

4. *When $\langle M,M\rangle$ is continuous, condition (ESC) and (SC) are equivalent. This will concern the applications to Sections 3. and 4.*

To obtain the solution to the minimization problem (2.1), we use Corollary 10 of [72] that we recall.

Corollary 2.2.24. *Under the assumption of Theorem 2.2.22, the solution of the minimization problem (2.1) is given by the pair $(H_0,\varphi^{(H_0)})$.*

In the sequel, we will find an explicit expression of the FS decomposition for a large class of square integrable random variables, when the underlying process is a process with independent increments, or is an exponential of process with independent increments. For this, the first step will consist in verifying (SC) and the boundedness condition on the MVT process, see Assumption 1.

2.2.5 Link with the equivalent signed martingale measure and the variance optimal martingale (VOM) measure

Definition 2.2.25. *1. A signed measure, Q, on $(\Omega, \mathcal{F}_T)$, is called a* **signed Θ-martingale measure**, *if*

(a) $Q(\Omega) = 1$;

(b) $Q \ll P$ *with* $\frac{dQ}{dP} \in \mathcal{L}^2(P)$;

(c) $\mathbb{E}[\frac{dQ}{dP} G_T(v)] = 0$ *for all* $v \in \Theta$.

We denote by $\mathbb{P}_s(\Theta)$, the set of all such signed Θ-martingale measures. Moreover, we define

$$\mathbb{P}_e(\Theta) := \{Q \in \mathbb{P}_s(\Theta) \mid Q \sim P \quad \textit{and } Q \textit{ is a probability measure}\} ,$$

and introduce the closed convex set,

$$\mathcal{D}_d := \{D \in \mathcal{L}^2(P) \mid D = \frac{dQ}{dP} \quad \textit{for some} \quad Q \in \mathbb{P}_s(\Theta)\} .$$

2. A signed martingale measure $\widetilde{P} \in \mathbb{P}_s(\Theta)$ is called variance-optimal martingale (VOM) measure if $\widetilde{D} = argmin_{D \in \mathcal{D}_d} Var[D^2] = argmin_{D \in \mathcal{D}_d} (\mathbb{E}[D^2] - 1)$, where $\widetilde{D} = \frac{d\widetilde{P}}{dP}$.

The space $G_T(\Theta) := \{G_T(v) \mid v \in \Theta\}$ is a linear subspace of $\mathcal{L}^2(P)$. Then, we denote by $G_T(\Theta)^{\perp}$ its orthogonal complement, that is,

$$G_T(\Theta)^{\perp} := \{D \in \mathcal{L}^2(P) \mid \mathbb{E}[DG_T(v)] = 0 \quad \text{for any } v \in \Theta\} .$$

Furthermore, $G_T(\Theta)^{\perp\perp}$ denotes the orthogonal complement of $G_T(\Theta)^{\perp}$, which is the $\mathcal{L}^2(P)$-closure of $G_T(\Theta)$.
A simple example when $\mathbb{P}_e(\Theta)$ is non empty is given by the following proposition, that anticipates some material treated in the next section.

Proposition 2.2.26. *Let X be a process with independent increments such that*

- *X_t has the same law as $-X_t$, for any $t \in [0, T]$;*
- *$\frac{1}{2}$ belongs to the domain D of the cumulative generating function $(t, z) \mapsto \kappa_t(z)$.*

Then, there is a probability $Q \sim P$ such that $S_t = \exp(X_t)$ is a martingale.

Proof. For all $t \in [0,T]$, we set $D_t = \exp\{-\frac{X_t}{2} - \kappa_t(-\frac{1}{2})\}$. Notice that D is a martingale so that the measure Q on $(\Omega, \mathcal{F}_T)$ defined by $dQ = D_T dP$ is an (equivalent) probability to P. On the other hand, the symmetry of the law of X_t implies for all $t \in [0,T]$,

$$S_t D_t = \exp\{\frac{X_t}{2} - \kappa_t(-\frac{1}{2})\} = \exp\{\frac{X_t}{2} - \kappa_t(\frac{1}{2})\} \ .$$

So SD is also a martingale. According to [53], chapter III, Proposition 3.8 a), S is a Q-martingale and so S is a Q-martingale. □

Example 2.2.27. *Let Y be a process with independent increments. We consider two copies Y^1 of Y and Y^2 of $-Y$. We set $X = Y^1 + Y^2$. Then X has the same law of $-X$.*

For simplicity, we suppose from now that Assumption 1 is verified, even if one could consider a more general framework, see [3] Therorem 1.28. This ensures that the linear space $G_T(\Theta)$ is closed in $\mathcal{L}^2(\Omega)$, therefore $G_T(\Theta) = \overline{G_T(\Theta)} = G_T(\Theta)^{\perp\perp}$. Moreover, Proposition 2.2.13 ensures that $\Theta = L^2(M)$. We recall an almost known fact cited in [3]. For completeness, we give a proof.

Proposition 2.2.28. $\mathbb{P}_s(\Theta) \neq \emptyset$ *is equivalent to* $1 \notin G_T(\Theta)$.

Proof. Let us prove the two implications.

- Let $Q \in \mathbb{P}_s(\Theta)$. If $1 \in G_T(\Theta)$, then $Q(\Omega) = \mathbb{E}^Q(1) = 0$ which leads to a contradiction since Q is a probability. Hence $1 \notin G_T(\Theta)$.

- Suppose that $1 \notin G_T(\Theta)$. We denote by f the orthogonal projection of 1 on $G_T(\Theta)$. Since $\mathbb{E}[f(1-f)] = 0$, then $\mathbb{E}[1-f] = \mathbb{E}[(1-f)^2]$. Recall that $1 \neq f \in G_T(\Theta)$, hence we have $\mathbb{E}[f] \neq 1$. Therefore, we can define the signed measure $\widetilde{P}$ by setting

$$\widetilde{P}(A) = \int_A \widetilde{D} dP \ , \quad \text{with} \quad \widetilde{D} = \frac{1-f}{1-\mathbb{E}[f]} \ . \tag{2.7}$$

  We check now that $\widetilde{P} \in \mathbb{P}_s(\Theta)$.

  - Trivially $\widetilde{P}(\Omega) = \mathbb{E}(\widetilde{D}) = 1$;
  - $\widetilde{P} \ll P$, by construction.
  - Let $v \in \Theta$, $\mathbb{E}[\widetilde{D} G_T(v)] = \frac{1}{1-\mathbb{E}[f]} \left(\mathbb{E}[(1-f)G_T(v)]\right) = 0$, since $1 - f \in G_T(\Theta)^{\perp}$.

Hence, $\widetilde{P} \in \mathbb{P}_s(\Theta)$ which concludes the proof of the Proposition.

$\square$

Remark 2.2.29. *If 1 is orthogonal to $G_T(\Theta)$, then $f = 0$ and $P \in \mathbb{P}_s(\Theta)$ so $\mathbb{P}_s(\Theta) \neq \emptyset$.*

In fact, $\widetilde{P}$ constructed in the proof of Proposition 2.2.28 coincides with the VOM measure.

Proposition 2.2.30. *Let $\widetilde{P}$ be the signed measure defined in (2.7). Then,*

$$\widetilde{D} = \arg\min_{D \in \mathcal{D}_d} \mathbb{E}[D^2] = \arg\min_{D \in \mathcal{D}_d} Var[D] \ .$$

Proof. Let $D \in \mathcal{D}_d$ and Q such that $dQ = DdP$. We have to show that $\mathbb{E}[D^2] \geq \mathbb{E}[\widetilde{D}^2]$. We write

$$\mathbb{E}[D^2] = \mathbb{E}[(D - \widetilde{D})^2] + \mathbb{E}[\widetilde{D}^2] + \frac{2}{1 - \mathbb{E}[f]} \mathbb{E}[(D - \widetilde{D})(1 - f)] \ .$$

Moreover, since $f \in G_T(\Theta)$ yields

$$\begin{aligned}
\mathbb{E}[(D - \widetilde{D})(1 - f)] &= \mathbb{E}[D] - \mathbb{E}[\widetilde{D}] - \mathbb{E}[Df] + \mathbb{E}[\widetilde{D}f] \ , \\
&= Q(\Omega) - \widetilde{Q}(\Omega) \ . \\
&= 0 \ .
\end{aligned}$$

$\square$

Remark 2.2.31.
1. *Arai [2] gives sufficient conditions under which the VOM measure is a probability, see Theorem 3.4 in [2].*
2. *Taking in account Proposition 2.2.28, the property $1 \notin G_T(\Theta)$ may be viewed as non-arbitrage condition. In fact, in [26], the existence of a martingale measure which is a probability is equivalent to a no free lunch condition.*

Next proposition can be easily deduced for a more general formulation, see [76].

Proposition 2.2.32. *We assume Assumption 1. Let $H \in \mathcal{L}^2(\Omega)$ and consider the solution (c^H, φ^H) of the minimization problem (2.1). Then, the price c^H equals the expectation under the VOM measure $\widetilde{P}$ of H.*

Proof. We have

$$H = c^H + G_T(\varphi^H) + R \ ,$$

where R is orthogonal to $G_T(\Theta)$ and $\mathbb{E}[R] = 0$. Since $\widetilde{P} \in \mathbb{P}_s(\Theta)$, taking the expectation with respect to $\widetilde{P}$, denoted by $\widetilde{\mathbb{E}}$ we obtain

$$\widetilde{\mathbb{E}}[H] = c^H + \widetilde{\mathbb{E}}[R] \ .$$

From the proof of Proposition 2.2.28, we have

$$\widetilde{\mathbb{E}}[R] = \frac{\mathbb{E}[(1-f)R]}{1-\mathbb{E}[f]} = \frac{1}{1-\mathbb{E}[f]}\left(\mathbb{E}[R] - \mathbb{E}[fR]\right) \ .$$

Since $f \in G_T(\Theta)$ and R is orthogonal to $G_T(\Theta)$, we get $\widetilde{\mathbb{E}}[R] = 0$. □

2.3 Processes with independent increments (PII)

This section deals with the case of Processes with Independent Increments. The preliminary part recalls some useful properties of such processes. Then, we obtain a sufficient condition on the characteristic function for the existence of the FS decomposition. Moreover, an explicit FS decomposition is derived.

Beyond its own theoretical interest, this work is motivated by its possible application to hedging and pricing energy derivatives and specifically electricity derivatives. Indeed, one way of modeling electricity forward prices is to use arithmetic models such as the Bachelier model which was developed for standard financial assets. The reason for using arithmetic models, is that the usual hedging intrument available on electricity markets are swap contracts which give a fixed price for the delivery of electricity over a contracted time period. Hence, electricty swaps can be viewed as a strip of forwards for each hour of the delivery period. In this framework, arithmetic models have the significant advantage to yield closed pricing formula for swaps which is not the case of geometric models.

However, in whole generality, an arithmetic model allows negative prices which could be underisable. Nevertheless, in the electricity market, negative prices may occur because it can be more expensive for a producer to switch off some generators than to pay someone to consume the resulting excess of production. Still, in [8], is introduced a class of arithmetic models where the positivity of spot prices is ensured, using a specific choice of increasing Lévy process. The parameters estimation of this kind of model is studied in [60].

2.3.1 Preliminaries

Definition 2.3.1. *$X = (X_t)_{t\in[0,T]}$ is a (real)* ***process with independent increments (PII)*** *iff*

1. *X has cadlag paths.*
2. *$X_0 = 0$.*
3. *$X_t - X_s$ is independent of $\mathcal{F}_s$ for $0 \le s < t \le T$ where $(\mathcal{F}_t)$ is the canonical filitration associated with X.*

   Moreover we will also suppose

4. *X is continuous in probability, i.e. X has no fixed time of discontinuties.*

From now on $(\mathcal{F}_t)$ will always be the canonical filtration associated with X. We recall Theorem II.4.15 of [53].

Theorem 2.3.2. *Let $(X_t)_{t\in[0,T]}$ be a real-valued special semimartingale, with $X_0 = 0$. Then, X is a process with independent increments, iff there is a version (b, c, ν) of its characteristics that is deterministic.*

Remark 2.3.3. *In particular, ν is a (deterministic non-negative) measure on the Borel σ-field of $[0, T] \times \mathbb{R}$.*

From now on, given two reals a, b, we denote by $a \vee b$ (resp. $a \wedge b$) the maximum (resp. minimum) between a and b.

Proposition 2.3.4. *Suppose X is a semimartingale with independent increments with characteristics (b, c, ν), then there exists an increasing function $t \mapsto a_t$ such that*

$$db_t \ll da_t \ , \quad dc_t \ll da_t \quad \text{and} \quad \nu(dt, dx) = \widetilde{F}_t(dx)da_t \ , \tag{3.1}$$

where $\widetilde{F}_t(dx)$ is a non-negative kernel from $\big([0,T], \mathcal{B}([0,T])\big)$ into $(\mathbb{R}, \mathcal{B})$ verifying

$$\int_{\mathbb{R}} (|x|^2 \wedge 1)\tilde{F}_t(dx) \le 1 \, , \quad \forall t \in [0, T]. \tag{3.2}$$

and

$$a_t = ||b||_t + c_t + \int_{\mathbb{R}} (|x|^2 \wedge 1)\nu([0,t], dx) \ . \tag{3.3}$$

Proof. The existence of (a_t) as a process fulfilling (3.3) and $\tilde{F}$ fulfilling (3.2) is provided by the statement and the proof of Proposition II. 2.9 of [53]. (3.3) and Theorem 2.3.2 guarantee that (a_t) is deterministic. $\square$

Remark 2.3.5. *In particular, (b_t), (c_t) and $t \mapsto \int_{[0,t]\times B}(|x|^2 \wedge 1)\nu(ds,dx)$ has bounded variation for any $B \in \mathcal{B}$.*

The proposition below provides the so called **Lévy-Khinchine Decomposition**.

Proposition 2.3.6. *Assume that $(X_t)_{t\in[0,T]}$ is a process with independent increments. Then*

$$\varphi_t(u) = e^{\Psi_t(u)} \ , \quad \textit{for all } u \in \mathbb{R} \ , \tag{3.4}$$

Ψ_t, is given by the Lévy-Khinchine decomposition of the process X,

$$\Psi_t(u) = iub_t - \frac{u^2}{2}c_t + \int_{\mathbb{R}}(e^{iux} - 1 - iuh(x))F_t(dx) \ , \quad \textit{for all } u \in \mathbb{R} \ , \tag{3.5}$$

where $B \mapsto F_t(B)$ is the positive measure $\nu([0,t] \times B)$ which integrates $1 \wedge |x|^2$ for any $t \in [0,T]$.

We introduce here a simplifying hypothesis for this section.

Assumption 2. *For any $t > 0$, X_t is never deterministic.*

Remark 2.3.7. *We suppose Assumption 2.*

1. *Up to a $2\pi i$ addition of $\kappa_t(e)$, we can write $\Psi_t(u) = \kappa_t(iu)$, $\forall u \in \mathbb{R}$. From now on we will always make use of this modification.*

2. *$\varphi_t(u)$ is never a negative number. Otherwise, there would be $u \in \mathbb{R}^*, t > 0$ such that $E(cos(uX_t)) = -1$. Since $cos(uX_t) + 1 \geq 0$ a.s. then $cos(uX_t) = -1$ a.s. and this is not possible since X_t is non-deterministic.*

3. *Previous point implies that all the differentiability properties of $u \mapsto \varphi_t(u)$ are equivalent to those of $u \mapsto \Psi_t(u)$.*

4. *If $\mathbb{E}[|X_t|^2] < \infty$, then for all $u \in \mathbb{R}$, $\Psi_t'(u)$ and $\Psi_t''(u)$ exist.*

We come back to the cumulant generating function κ and its domain D.

Remark 2.3.8. *In the case where the underlying process is a PII, then*

$$D := \{z \in \mathbb{C} \mid \mathbb{E}[e^{Re(z)X_t}] < \infty, \ \forall t \in [0,T]\} = \{z \in \mathbb{C} \mid \mathbb{E}[e^{Re(z)X_T}] < \infty\} \ .$$

In fact, for given $t \in [0,T], \gamma \in \mathbb{R}$ we have

$$\mathbb{E}(e^{\gamma X_T}) = \mathbb{E}(e^{\gamma X_t})\mathbb{E}(e^{\gamma(X_T - X_t)}) < \infty.$$

Since each factor is positive, and if the left-hand side is finite, then $\mathbb{E}(e^{\gamma X_t})$ is also finite.

We need now a result which extends the Lévy-Khinchine decomposition to the cumulant generating function. Similarly to Theorem 25.17 of [69] we have.

Proposition 2.3.9. *Let $D_0 = \left\{c \in \mathbb{R} \mid \int_{[0,T]\times\{|x|>1\}} e^{cx}\nu(dt,dx) < \infty\right\}$. Then,*

1. *D_0 is convex and contains the origin.*
2. *$D_0 = D \cap \mathbb{R}$.*
3. *If $z \in \mathbb{C}$ such that $Re(z) \in D_0$, i.e. $z \in D$, then*

$$\kappa_t(z) = zb_t + \frac{z^2}{2}c_t + \int_{[0,t]\times\mathbb{R}} (e^{zx} - 1 - zh(x))\nu(ds,dx) \ . \tag{3.6}$$

Proof. 1. is a consequence of Hölder inequality similarly as i) in Theorem 25.17 of [69] .

2. The characteristic function of the law of X_t is given by (3.5). According to Theorem II.8.1 (iii) of Sato [69], there is an infinitely divisible distribution with characteristics $(b_t, c_t, F_t(dx))$, fulfilling $F_t(\{0\}) = 0$ and $\int(1 \wedge x^2)F_t(dx) < \infty$ and $c_t \geq 0$. By uniqueness of the characteristic function, that law is precisely the law of X_t. By Corollary II.11.6, in [69], there is a Lévy process $(L_s^t, 0 \leq s \leq 1)$ such that L_1^t and X_t are identically distributed. We define

$$C_0^t = \{c \in \mathbb{R} \mid \int_{\{|x|>1\}} e^{cx}F_t(dx) < \infty\} \quad \text{and} \quad C^t = \{z \in \mathbb{C} \mid \mathbb{E}\left[\exp(Re(zL_1^t)\right] < \infty\} \ .$$

   Remark 2.3.8 says that $C^T = D$, moreover clearly $C_0^T = D_0$. Theorem V.25.17 of [69] implies $D_0 = D \cap \mathbb{R}$, i.e. point 2. is established.

3. Let $t \in [0,T]$ be fixed; let $w \in D$. We apply point (iii) of Theorem V.25.17 of [69] to the Lévy process L^t.

□

Proposition 2.3.10. *Let X be a semimartingale with independent increments. For all $z \in D$, $t \mapsto \kappa_t(z)$ has bounded variation and*

$$\kappa_{dt}(z) \ll da_t \ . \tag{3.7}$$

Proof. Using (3.6), it remains to prove that

$$t \mapsto \int_{[0,T]\times\mathbb{R}} (e^{zx} - 1 - zh(x))\nu(ds, dx)$$

is absolutely continuous with respect to (da_t). We can conclude

$$\kappa_t(z) = \int_0^t \frac{db_s}{da_s} da_s + \frac{z^2}{2}\int_0^t \frac{dc_s}{da_s} da_s + \int_0^t da_s \int_{\mathbb{R}} (e^{zx} - 1 - zh(x))\, \widetilde{F}_s(dx)\ ,$$

if we show that

$$\int_0^T da_s \int_{\mathbb{R}} |e^{zx} - 1 - zh(x)| \widetilde{F}_s(dx) < \infty\ . \tag{3.8}$$

Without restriction of generality we can suppose $h(x) = x1_{|x|\leq 1}$. (3.8) can be bounded by the sum $I_1 + I_2 + I_3$ where

$$\begin{aligned}
I_1 &= \int_0^T da_s \int_{|x|>1} |e^{zx}| \widetilde{F}_s(dx) \\
I_2 &= \int_0^T da_s \int_{|x|>1} \widetilde{F}_s(dx) \\
I_3 &= \int_0^T da_s \int_{|x|\leq 1} |e^{zx} - 1 - zx| \widetilde{F}_s(dx)
\end{aligned}$$

Using Proposition 2.3.4, we have

$$I_1 = \int_0^T da_s \int_{|x|>1} |e^{zx}| \widetilde{F}_s(dx) = \int_0^T da_s \int_{|x|>1} |e^{Re(z)x}| \widetilde{F}_s(dx) = \int_{[0,T]\times|x|>1} |e^{Re(z)x}| \nu(ds, dx);$$

this quantity is finite because $Re(z) \in D_0$ taking into account Proposition 2.3.9. Concerning I_2 we have

$$I_2 = \int_0^T da_s \int_{|x|>1} \widetilde{F}_s(dx) = \int_0^T da_s \int_{|x|>1} (1 \wedge |x^2|) \widetilde{F}_s(dx) \leq a_T,$$

because of (3.2). As far as I_3 is concerned, we have

$$I_3 \leq e^{Re(z)} \frac{z^2}{2} \int_{[0,T]\times|x|\leq 1} da_s(x^2 \wedge 1)\widetilde{F}_s(dx) = e^{Re(z)} \frac{z^2}{2} a_T$$

again because of (3.2). This concludes the proof of the Proposition.

□

The converse of the first part of previous corollary also holds. For this purpose we formulate first a simple remark.

Remark 2.3.11. *For every $z \in D$, $(\exp(zX_t - \kappa_t(z)))$ is a martingale. In fact, for all $0 \leq s \leq t \leq T$, we have*

$$\mathbb{E}[\exp(z(X_t - X_s))] = \exp(\kappa_t(z) - \kappa_s(z)) \ . \tag{3.9}$$

Proposition 2.3.12. *Let X be a PII. Let $z \in D \cap \mathbb{R}^\star$. $(X_t)_{t\in[0,T]}$ is a semimartingale iff $t \mapsto \kappa_t(z)$ has bounded variation.*

Proof. It remains to prove the converse implication.
If $t \mapsto \kappa_t(z)$ has bounded variation then $t \mapsto e^{\kappa_t(z)})$ has the same property. Remark 2.3.11 says that $e^{zX_t} = M_t e^{\kappa_t(z)}$ where (M_t) is a martingale. Finally, (e^{zX_t}) is a semimartingale and taking the logarithm (zX_t) has the same property. □

Remark 2.3.13. *Let $z \in D$. If (X_t) is a semimartingale with independent increments then (e^{zX_t}) is necessarily a special semimartingale since it is the product of a martingale and a bounded variation continuous deterministic function, by use of integration by parts.*

Lemma 2.3.14. *Suppose that (X_t) is a semimartingale with independent increments. Then for every $z \in Int(D)$, $t \mapsto \kappa_t(z)$ is continuous.*

Remark 2.3.15. *The conclusion remains true for any process which is continuous in probability, whenever $t \mapsto \kappa_t(z)$ is (locally) bounded.*

Proof of Lemma 2.3.14. Since $z \in Int(D)$, there is $\gamma > 1$ such that $\gamma z \in D$; so

$$\mathbb{E}[\exp(z\gamma X_t)] = \exp(\kappa_t(\gamma z)) \leq \exp(\sup_{t\leq T}(\kappa_t(\gamma z))) \ ,$$

because $t \mapsto \kappa_t(\gamma z)$ is bounded, being of bounded variation. This implies that $(\exp(zX_t))_{t\in[0,T]}$ is uniformly integrable. Since (X_t) is continuous in probability, then $(\exp(zX_t))$ is continuous in $\mathcal{L}^1$. The result easily follows. □

Proposition 2.3.16. *The function $(t,z) \mapsto \kappa_t(z)$ is continuous. In particular, $(t,z) \mapsto \kappa_t(z)$, $t \in [0,T]$, z belonging to a compact real subset, is bounded.*

Proof.

- Proposition 2.3.9 implies that $z \mapsto \kappa_t(z)$ is continuous uniformly with respect to $t \in [0,T]$.
- By Lemma 2.3.14, for $z \in \mathrm{IntD}$, $t \mapsto \kappa_t(z)$ is continuous.
- To conclude it is enough to show that $t \mapsto \kappa_t(z)$ is continuous for every $z \in D$. Since $\bar{D} = \overline{\mathrm{IntD}}$, there is a sequence (z_n) in the interior of D converging to z. Since a uniform limit of continuous functions on $[0,T]$ converges to a continuous function, the result follows.

□

2.3.2 Structure condition for PII (which are semimartingales)

Let $X = (X_t)_{t\in[0,T]}$ be a real-valued semimartingale with independent increments and $X_0 = 0$. We assume that $\mathbb{E}[|X_t|^2] < \infty$. We denote by $\varphi_t(u) = \mathbb{E}[\exp(iuX_t)]$ the characteristic function of X_t and by $u \mapsto \Psi_t(u)$ its log-characteristic function introduced in Proposition 2.3.6. We recall that $\varphi_t(u) = \exp(\Psi_t(u))$.
X has the property of independent increments; therefore

$$\exp(iuX_t)/\mathbb{E}[\exp(iuX_t)] = \exp(iuX_t)/\exp(\Psi_t(u)) \ , \tag{3.10}$$

is a martingale.

Remark 2.3.17. *Notice that the two first order moments of X are related to the log-characterisctic function of X, as follows*

$$\begin{aligned} \mathbb{E}[X_t] &= -i\Psi_t^{'}(0) \ , \quad \mathbb{E}[X_t - X_s] = -i(\Psi_t^{'}(0) - \Psi_s^{'}(0)), && (3.11)\\ Var(X_t) &= -\Psi_t^{''}(0) \ , \quad Var(X_t - X_s) = -[\Psi_t^{''}(0) - \Psi_s^{''}(0)] \ . && (3.12) \end{aligned}$$

Proposition 2.3.18. *Let $X = (X_t)_{t\in[0,T]}$ be a real-valued semimartingale with independent increments.*

1. *X is a special semimartingale with decomposition $X = M + A$ with the following properties:*

$$\langle M \rangle_t = -\Psi_t^{''}(0) \quad \text{and} \quad A_t = -i\Psi_t^{'}(0) \ . \tag{3.13}$$

   In particular $t \mapsto -\Psi_t^{''}(0)$ is increasing and therefore of bounded variation.

2. *X satisfies condition (SC) of Definition 2.2.10 if and only if*

$$\Psi_t^{'}(0) \ll \Psi_t^{''}(0) \quad \textit{and} \quad \int_0^T \left| \frac{d_t\Psi_s^{'}}{d_t\Psi_s^{''}}(0) \right|^2 |d\Psi_s^{''}(0)| < \infty \ . \tag{3.14}$$

In that case

$$A_t = \int_0^t \alpha_s d\langle M \rangle_s \quad \textit{with} \quad \alpha_t = i\frac{d_t\Psi_t^{'}(0)}{d_t\Psi_t^{''}(0)} \quad \textit{for all } t \in [0,T]. \tag{3.15}$$

3. *Under condition (3.14), FS decomposition exists (and it is unique) for every square integrable random variable.*

In the sequel, we will provide an explicit decomposition for a class of contingent claims, under condition (3.14).

Proof. 1. Let us first determine A and M in terms of the log-characteristic function of X. Using (3.11) of Remark 2.3.17, we get

$$\begin{aligned}
\mathbb{E}[X_t|\mathcal{F}_s] &= \mathbb{E}[X_t - X_s + X_s \mid \mathcal{F}_s] \ , = \mathbb{E}[X_t - X_s] + X_s \ , \\
&= -i\Psi_t^{'}(0) + i\Psi_s^{'}(0) + X_s \ , \quad \text{then} \ , \\
\mathbb{E}[X_t + i\Psi_t^{'}(0)|\mathcal{F}_s] &= X_s + i\Psi_s^{'}(0) \ .
\end{aligned}$$

Hence, $(X_t + i\Psi_t^{'}(0))$ is a martingale and the canonical decomposition of X follows

$$X_t = \underbrace{X_t + i\Psi_t^{'}(0)}_{M_t} \underbrace{-i\Psi_t^{'}(0)}_{A_t} \ ,$$

where M is a local martingale and A is a locally bounded variation process thanks to the semimartingale property of X. Let us now determine $\langle M \rangle$, in terms of the log-characteristic function of X. Using (3.11) and (3.12) of Remark 2.3.17, yields

$$\begin{aligned}
\mathbb{E}[M_t^2|\mathcal{F}_s] &= \mathbb{E}[(X_t + i\Psi_t^{'}(0))^2|\mathcal{F}_s] = \mathbb{E}[(M_s + X_t - X_s + i(\Psi_t^{'}(0) - \Psi_s^{'}(0)))^2|\mathcal{F}_s] \ , \\
&= M_s^2 + Var(X_t - X_s) = M_s^2 - \Psi_t^{''}(0) + \Psi_s^{''}(0) \ .
\end{aligned}$$

Hence, $(M_t^2 + \Psi_t^{''}(0))$ is a $(\mathcal{F}_t)$-martingale, and point 1. is established. On the other hand

$$A_t = \int_0^t \alpha_s d\langle M \rangle_s \quad \text{with} \quad \alpha_t = i\frac{d_t\Psi_t^{'}(0)}{d_t\Psi_t^{''}(0)} \quad \text{for all } t \in [0,T] \ .$$

2. is a consequence of point 1. and of Definition 2.2.10.

3. follows from Theorem 2.2.19. In fact $K_T = -\int_0^T \left(\frac{d_t\Psi_s^{'}}{d_t\Psi_s^{''}}(0)\right)^2 d\Psi_s^{''}(0)$ is deterministic and so Assumption 1 is fulfilled.

□

2.3.3 Examples

A Gaussian continuous process example

Let $\psi : [0,T] \to \mathbb{R}$ be a continuous strictly increasing function, $\gamma : [0,T] \to \mathbb{R}$ be a bounded variation function such that $d\gamma \ll d\psi$. We set $X_t = W_{\psi(t)} + \gamma(t)$, where W is the standard Brownian motion on $\mathbb{R}$. Clearly, $X_t = M_t + \gamma(t)$, where $M_t = W_{\psi(t)}$, defines a continuous martingale, such that $\langle M\rangle_t = [M]_t = \psi(t)$. Since $X_t \sim \mathcal{N}(\gamma(t), \psi(t))$ for all $u \in \mathbb{R}$ and $t \in [0,T]$, we have

$$\Psi_t(u) = i\gamma(t)u - \frac{u^2\psi(t)}{2} ,$$

which yields

$$\Psi_t^{'}(0) = i\gamma(t) \quad \text{and} \quad \Psi_t^{''}(0) = -\psi(t) ,$$

Therefore, if $\frac{d\gamma}{d\psi} \in \mathcal{L}^2(d\psi)$, then X satisfies condition (SC) of Definition 2.2.10 with

$$A_t = \int_0^t \alpha_s d\langle M\rangle_s \quad \text{and} \quad \alpha_t = \left.\frac{d\gamma}{d\psi}\right|_t \quad \text{for all } t \in [0,T] .$$

Processes with independent and stationary increments (Lévy processes)

Definition 2.3.19. *$X = (X_t)_{t\in[0,T]}$ is called* **Lévy process** *or process with stationary and independent increments if X is a PII process such that the distribution of $X_t - X_s$ depends only on $t-s$ for $0 \le s \le t \le T$.*

For details on Lévy processes, we refer the reader to [63], [69] and [53].
Let $X = (X_t)_{t\in[0,T]}$ be a real-valued Lévy process, with $X_0 = 0$. We assume that $\mathbb{E}[|X_t|^2] < \infty$ and we do not consider the trivial case where L_1 is deterministic.

Remark 2.3.20. *1. Since $X = (X_t)_{t\in[0,T]}$ is a Lévy process then $\Psi_t(u) = t\Psi_1(u)$. In the sequel, we will use the shortened notation $\Psi := \Psi_1$.*

2. *Ψ is a function of class C^2 and $\Psi^{''}(0) = Var(X_1)$ which is strictly positive if X has no stationary increments.*

We recall some cumulant and log-caracteristic functions of some typical Lévy processes.

Remark 2.3.21. 1. *Poisson Case: If X is a Poisson process with intensity λ, we have that $\kappa^\Lambda(z) = \lambda(e^z - 1)$. Moreover, in this case the set $D = \mathbb{C}$.*
Concerning the log-characteristic function we have

$$\Psi(u) = \lambda(e^{iu} - 1) \ , \quad \Psi^{'}(0) = i\lambda \quad \textit{and} \quad \Psi^{''}(0) = -\lambda, u \in \mathbb{R}.$$

2. *NIG Case: This process was introduced by Barndorff-Nielsen in [6]. Then X is a Lévy process with $X_1 \sim NIG(\alpha, \beta, \delta, \mu)$, with $\alpha > |\beta| > 0$, $\delta > 0$ and $\mu \in \mathbb{R}$. We have $\kappa^\Lambda(z) = \mu z + \delta(\gamma_0 - \gamma_z)$ and $\gamma_z = \sqrt{\alpha^2 - (\beta + z)^2}$, $D = [-\alpha - \beta, \alpha - \beta] + i\mathbb{R}$.*
Therefore

$$\Psi(u) = \mu i u + \delta(\gamma_0 - \gamma_{iu}) \ , \quad \textit{where} \quad \gamma_{iu} = \sqrt{\alpha^2 - (\beta + iu)^2} \ .$$

By derivation, one gets

$$\Psi^{'}(0) = i\mu + \delta\frac{i\beta}{\gamma_0} \quad \textit{and} \quad \Psi^{''}(0) = -\delta(\frac{1}{\gamma_0} + \frac{\beta^2}{\gamma_0^3}),$$

Which yields $\alpha = i\dfrac{\Psi^{'}(0)}{\Psi^{''}(0)} = \dfrac{\gamma_0^2(\gamma_0\mu + \delta\beta)}{\delta(\gamma_0^2 + \beta)}$.

3. *Variance Gamma case: Let $\alpha, \beta > 0, \delta \neq 0$. If X is a Variance Gamma process with $X_1 \sim VG(\alpha, \beta, \delta, \mu)$ with $\kappa^\Lambda(z) = \mu z + \delta Log\left(\frac{\alpha}{\alpha - \beta z - \frac{z^2}{2}}\right)$, where Log is again the principal value complex logarithm defined in Section 2. The expression of $\kappa^\Lambda(z)$ can be found in [49, 59] or also [22], table IV.4.5 in the particular case $\mu = 0$. In particular an easy calculation shows that we need $z \in \mathbb{C}$ such that $Re(z) \in]-\beta - \sqrt{\beta^2 + 2\alpha}, -\beta + \sqrt{\beta^2 + 2\alpha}[$ so that $\kappa^\Lambda(z)$ is well defined so that*

$$D =]-\beta - \sqrt{\beta^2 + 2\alpha}, -\beta + \sqrt{\beta^2 + 2\alpha}[+ i\mathbb{R}.$$

Finally we obtain

$$\Psi(u) = \mu i u + \delta Log\left(\frac{\alpha}{\alpha - \beta i u + \frac{u^2}{2}}\right).$$

After derivation it follows

$$\Psi^{'}(0) = i(\mu - \delta\beta), \quad \Psi^{''}(0) = \frac{\delta}{\alpha}(\alpha^2 - \beta^2).$$

We discuss now the validity of the (SC) in the Lévy case. By application of Proposition 2.3.18 and Remark 2.3.20, we get the following result.

Corollary 2.3.22. *Let $X = M + A$ be the canonical decomposition of X, then for all $t \in [0, T]$,*

$$\langle M \rangle_t = -t\Psi''(0) \quad \textit{and} \quad A_t = -it\Psi'(0) \ . \tag{3.16}$$

Moreover X satisfies condition (SC) of Definition 2.2.10 with

$$A_t = \int_0^t \alpha d \langle M \rangle_s \quad \textit{with} \quad \alpha = i\frac{\Psi'(0)}{\Psi''(0)} \quad \textit{for all } t \in [0, T] \ . \tag{3.17}$$

Hence, FS decomposition exists for every square integrable random variable.

Remark 2.3.23. *We have the following in previous three examples in Remark 2.3.21.*

1. **Poisson case:** $\alpha = 1$.
2. **NIG process:** $\alpha = \dfrac{\gamma_0^2(\gamma_0\mu + \delta\beta)}{\delta(\gamma_0^2 + \beta)}$.
3. **VG process:** $\alpha = \dfrac{\mu - \delta\beta}{\alpha^2 - \beta^2}\dfrac{\alpha}{\delta}$.

Wiener integrals of Lévy processes

We take $X_t = \int_0^t \gamma_s d\Lambda_s$, where Λ is a square integrable Lévy process as in Section 2.3.3. Then, $\int_0^T \gamma_s d\Lambda_s$ is well-defined for at least $\gamma \in \mathcal{L}^\infty([0, T])$. It is then possible to calculate the characteristic function and the cumulative function of $\int_0^\cdot \gamma_s d\Lambda_s$. Let $(t, z) \mapsto t\Psi_\Lambda(z)$, (resp. $(t, z) \mapsto t\kappa^\Lambda(z)$) denoting the log-characteristic function (resp. the cumulant generating function) of Λ.

Lemma 2.3.24. *Let $\gamma : [0, T] \to \mathbb{R}$ be a Borel bounded function.*

1. *The log-characteristic function of X_t is such that for all $u \in \mathbb{R}$,*
$$\Psi_{X_t}(u) = \int_0^t \Psi_\Lambda(u\gamma_s)ds \ , \quad \textit{where} \quad \mathbb{E}[\exp(iuX_t)] = \exp\left(\Psi_{X_t}(u)\right) \ ;$$
2. *Let D_Λ be the domain related to κ^Λ in the sense of Definition 2.2.2. The cumulant generating function of X_t is such that for all $z \in \{z \mid Re z\gamma_t \in D_\Lambda \text{ for all } t \in [0, T]\}$,*
$$\kappa_{X_t}(z) = \int_0^t \kappa^\Lambda(z\gamma_s)ds.$$

Proof. We only prove 1. since 2. follows similarly. Suppose first γ to be continuous, then $\int_0^T \gamma_s d\Lambda_s$ is the limit in probability of $\sum_{j=0}^{p-1} \gamma_{t_j}(\Lambda_{t_{j+1}} - \Lambda_{t_j})$ where $0 = t_0 < t_1 < ... < t_p = T$ is a subdivision of $[0, T]$ whose mesh converges to zero. Using the independence of the increments, we have

$$\begin{aligned}
\mathbb{E}\left[\exp\{i\sum_{j=0}^{p-1}\gamma_{t_j}(\Lambda_{t_{j+1}} - \Lambda_{t_j})\}\right] &= \prod_{j=0}^{p-1}\mathbb{E}\left[\exp\{i\gamma_{t_j}(\Lambda_{t_{j+1}} - \Lambda_{t_j})\}\right] , \\
&= \prod_{j=0}^{p-1}\exp\{\Psi_\Lambda(\gamma_{t_j})(t_{j+1} - t_j)\} , \\
&= \exp\{\sum_{j=0}^{p-1}(t_{j+1} - t_j)\Psi_\Lambda(\gamma_{t_j})\} .
\end{aligned}$$

This converges to $\exp\left(\int_0^T \Psi_\Lambda(\gamma_s)ds\right)$, when the mesh of the subdivision goes to zero. Suppose now that γ is only bounded and consider, using convolution, a sequence γ_n of continuous functions, such that $\gamma_n \to \gamma$ a.e. and $\sup_{t\in[0,T]} |\gamma_n(t)| \le \sup_{t\in[0,T]} |\gamma(t)|$. We have proved that

$$\mathbb{E}\left[\exp\left(i\int_0^T \gamma_n(s)d\Lambda_s\right)\right] = \exp\left(\int_0^T \Psi_\Lambda(\gamma_n(s))ds\right) \tag{3.18}$$

Now, Ψ_Λ is continuous therefore bounded, so Lebesgue dominated convergence and continuity of stochastic integral imply statement 1. □

Remark 2.3.25.
1. *Previous proof, which is left to the reader, also applies for statement 2. This statement in a slight different form is proved in [11]*
2. *We prefer to formulate a direct proof. In particular statement 1. holds with the same proof even if Λ has no moment condition and γ is a continuous function with bounded variation. Stochastic integrals are then defined using integration by parts.*

We suppose now that Λ is a Lévy process such that Λ_1 is not deterministic. In particular $Var(\Lambda_1) \neq 0$ and so $\Psi''_\Lambda \neq 0$.
In this case

$$\Psi'_t(u) = \int_0^t \Psi'_\Lambda(u\gamma_s)\gamma_s ds \quad \text{and} \quad \Psi''_t(u) = \int_0^t \Psi''_\Lambda(u\gamma_s)\gamma_s^2 ds .$$

So

$$\Psi'_t(0) = \Psi'_\Lambda(0) \int_0^t \gamma_s ds \quad \text{and} \quad \Psi''_t(0) = \Psi''_\Lambda(0) \int_0^t \gamma_s^2 ds \ .$$

Condition (SC) is verified since $d\Psi'_t(0) \ll d\Psi''_t(0)$ with

$$\alpha_t = i\frac{d\Psi'_t(0)}{d\Psi''_t(0)} = \frac{\Psi'_\Lambda(0)}{\Psi''_\Lambda(0)} \frac{i}{\gamma_t} 1_{\{\gamma_t \neq 0\}} \quad \text{and} \quad \int_0^T \alpha_s^2 \, |\Psi''_s(0)| \gamma_s^2 ds = T\frac{|\Psi'_\Lambda(0)|^2}{|\Psi''_\Lambda(0)|} < \infty \ .$$

2.3.4 Explicit Föllmer-Schweizer decomposition in the PII case

Preliminaries

Let $X = (X_t)_{t\in[0,T]}$ be a semimartingale with independent increments with log-characteristic function $(t, u) \mapsto \Psi_t(u)$. We assume that $(X_t)_{t\in[0,T]}$ is square integrable and satisfies Assumption 2.

Remark 2.3.26. *1. $u \mapsto \Psi_t(u)$ is of class C^2, for any $t \in [0, T]$ because X_t is square integrable.*

2. $t \mapsto \Psi''_t(0)$ and $t \mapsto \Psi'_t(0)$ have bounded variation because of Proposition 2.3.18. Therefore, they are bounded.

3. $t \mapsto \Psi'_t(u)$ is continuous for every $u \in \mathbb{R}$. In fact, first $t \mapsto X_t$ is continuous in probability. Since $M_t = X_t - \Psi'_t(0)$ is a square integrable martingale and $t \mapsto \Psi'_t(0)$ is bounded, then the family $(E(X_t^2))$ is bounded and so (X_t) is uniformly integrable. So $t \mapsto \varphi'_t(u)$ is continuous and the result follows by Assumption 2

4. $t \mapsto \Psi''_t(0)$ is continuous. In fact, again it is enough to prove $t \mapsto \varphi''_t(0)$ is continuous. This follows if we prove that (M_t) is continuous in $\mathcal{L}^2$. This is true because M is continuous in probability and for any $N > 0$, $t \in [0, T]$, Chebyshev implies that

$$P\{|M_t^2| > N\} \le \frac{\mathrm{Var}(X_t)}{N} \le \frac{\mathrm{Var}(X_T)}{N},$$

and so the family (M_t^2) is again uniformly integrable.

We suppose the following.

Assumption 3. *1. $t \mapsto \Psi'_t(u)$ is absolutely continuous with respect to $d\Psi''_t(0)$.*

2. *For every $u \in \mathbb{R}$, we suppose that the following quantity*

$$K(u) := \int_0^T \left| \frac{d\Psi_t'(u)}{d\Psi_t''(0)} \right|^2 \exp(2Re(\Psi_T(u) - \Psi_t(u)))d(-\Psi_t''(0)) \tag{3.19}$$

is finite.

Remark 2.3.27. *If $u = 0$, the previous quantity* (3.19) *is finite because of the (SC) condition.*

We consider a contingent claim which is given as a Fourier transform of X_T,

$$H = f(X_T) \quad \text{with} \quad f(x) = \int_{\mathbb{R}} e^{iux}\mu(du) \ , \quad \text{for all } x \in \mathbb{R} \ , \tag{3.20}$$

for some finite signed measure μ.

Assumption 4.

$$\int_{\mathbb{R}} K(u)d|\mu(u)| < \infty.$$

Remark 2.3.28. *We observe that the function*

$$\begin{array}{rcl} (u,t) & \mapsto & \exp(\Psi_T(u) - \Psi_t(u)) \\ (u,t) & \mapsto & \exp(2(\Psi_T(u) - \Psi_t(u))) \end{array}$$

are uniformly bounded because the characteristic function is bounded.

We will first evaluate an explicit Kunita-Watanabe decomposition of H w.r.t. the martingale part M of X. Later, we will finally obtain the decomposition with respect to X.

Explicit elementary Kunita-Watanabe decomposition

By Propostion 2.3.18, X admits the following semimartingale decomposition, $X_t = A_t + M_t$, where

$$A_t = -i\Psi_t'(0) \quad \text{and} \quad \langle M \rangle_t = -\Psi_t''(0) \ . \tag{3.21}$$

Proposition 2.3.29. *Let $H = f(X_T)$ where f is of the form (3.20). We suppose that the PII X satisfies Assumptions 2, 3 and 4. Then, H admits the decomposition*

$$\left\{ \begin{array}{rcl} V_t & = & V_0 + \int_0^t Z_s dM_s + O_t \\ V_T & = & H \ , \end{array} \right. \tag{3.22}$$

with the following properties.

1. *$H = V_T$ where $(V_t)_{t\in[0,T]}$ is an $(\mathcal{F}_t)$-martingale defined by*

$$V_t = \mathbb{E}(H|\mathcal{F}_t) = \int_{\mathbb{R}} V_t(u)d\mu(u), \quad t \in [0,T],$$

where for any $u \in \mathbb{R}$ we have

$$V_t(u) = e^{iuX_t} \exp\left\{\Psi_T(u) - \Psi_t(u)\right\}. \tag{3.23}$$

2. *For all $t \in [0,T]$, $\quad Z_t = \int_{\mathbb{R}} Z_t(u)d\mu(u)$ where for any $u \in \mathbb{R},\ t \in [0,T]$*

$$Z_t(u) = ie^{iuX_{t-}} \frac{d\left(\Psi_t'(u) - \Psi_t'(0)\right)}{d\Psi_t''(0)} \exp\left\{\Psi_T(u) - \Psi_t(u)\right\}; \tag{3.24}$$

3. $\mathbb{E}\left[\int_0^T Z_s^2 d\left\langle M\right\rangle_s\right] < \infty\,.$

4. *O is a square integrable $(\mathcal{F}_t)$-martingale such that $\langle O, M\rangle = 0$.*

Remark 2.3.30. *In particular, $V_0 = \mathbb{E}[H]$.*

Proof. A) We start with the case $\mu = \delta_u(dx)$ for some $u \in \mathbb{R}$ so that $f(x) = e^{iux}$. We consider the $(\mathcal{F}_t)$-martingale $V_t = \mathbb{E}[f(X_T)|\mathcal{F}_t] = \mathbb{E}[e^{iuX_T}|\mathcal{F}_t]$.

1. Clearly $V_0 = \mathbb{E}[e^{iuX_T}]$.
2. We calculate explicitely V_t, which gives

$$\begin{aligned} V_t &= \mathbb{E}[e^{iuX_T}|\mathcal{F}_t] \ = e^{iuX_t}\mathbb{E}[e^{iu(X_T-X_t)}] \ = \exp(iuX_t - \Psi_t(u))\exp(\Psi_T(u)) \\ &= \widetilde{V}_t \exp(\Psi_T(u))\ , \end{aligned}$$

where $\widetilde{V}_t = \exp(iuX_t - \Psi_t(u))$ defines an $(\mathcal{F}_t)$-martingale.

3. We evaluate $\langle V, M\rangle$.

Lemma 2.3.31. $\langle V, M\rangle_t = -i\int_0^t V_s(\Psi'_{ds}(u) - \Psi'_{ds}(0))\ .$

Proof. We evaluate $\mathbb{E}[\widetilde{V}_t M_t|\mathcal{F}_s]$. Since $\widetilde{V}$ and M are $(\mathcal{F}_t)$-martingales and using the property of independent increments we get

$$\begin{aligned} \mathbb{E}[\widetilde{V}_t M_t|\mathcal{F}_s] &= \mathbb{E}[\widetilde{V}_t M_s|\mathcal{F}_s] + \mathbb{E}[\widetilde{V}_t(M_t - M_s)|\mathcal{F}_s]\ , \\ &= M_s\widetilde{V}_s + \widetilde{V}_s\mathbb{E}[\exp\{iu(X_t - X_s) - (\Psi_t(u) - \Psi_s(u))\}(M_t - M_s)]\ , \\ &= M_s\widetilde{V}_s + \widetilde{V}_s e^{-(\Psi_t(u)-\Psi_s(u))}\mathbb{E}[e^{iu(X_t-X_s)}(M_t - M_s)]\ . \end{aligned}$$

Previous expectation gives

$$\begin{aligned}
\mathbb{E}[e^{iu(X_t-X_s)}(M_t-M_s)] &= \mathbb{E}[e^{iu(X_t-X_s)}(X_t-X_s)]+\mathbb{E}[e^{iu(X_t-X_s)}i(\Psi_t^{'}(0)-\Psi_s^{'}(0))] \ , \\
&= -i\frac{\partial}{\partial u}\mathbb{E}[e^{iu(X_t-X_s)}]+i(\Psi_t^{'}(0)-\Psi_s^{'}(0))\mathbb{E}[e^{iu(X_t-X_s)}] \ , \\
&= -ie^{\Psi_t(u)-\Psi_s(u)}(\Psi_t^{'}(u)-\Psi_s^{'}(u))+i(\Psi_t^{'}(0)-\Psi_s^{'}(0))e^{\Psi_t(u)-\Psi_s(u)} \ .
\end{aligned}$$

Consequently,

$$\begin{aligned}
\mathbb{E}[\widetilde{V}_tM_t|\mathcal{F}_s] &= M_s\widetilde{V}_s-i\widetilde{V}_s(\Psi_t^{'}(u)-\Psi_s^{'}(u))+i\widetilde{V}_s(\Psi_t^{'}(0)-\Psi_s^{'}(0)) \\
&= M_s\widetilde{V}_s-i\widetilde{V}_s\left(\Psi_t^{'}(u)-\Psi_t^{'}(0)-(\Psi_s^{'}(u)-\Psi_s^{'}(0))\right) \ .
\end{aligned}$$

This implies that $\left(\widetilde{V}_tM_t+i\widetilde{V}_t(\Psi_t^{'}(u)-\Psi_t^{'}(0))\right)_t$ is an $(\mathcal{F}_t)$-martingale. Then by integration by parts,

$$\widetilde{V}_t(\Psi_t^{'}(u)-\Psi_t^{'}(0))=\int_0^t\widetilde{V}_s(\Psi_{ds}^{'}(u)-\Psi_{ds}^{'}(0))+\int_0^t(\Psi_s^{'}(u)-\Psi_s^{'}(0))d\widetilde{V}_s \ .$$

The second integral term of the right-hand side being a martingale, it follows that

$$\left\langle\widetilde{V},M\right\rangle_t=-i\int_0^t\widetilde{V}_s(\Psi_{ds}^{'}(u)-\Psi_{ds}^{'}(0)) \ .$$

and so

$$\langle V,M\rangle_t=-i\int_0^tV_s(\Psi_{ds}^{'}(u)-\Psi_{ds}^{'}(0)) \ . \tag{3.25}$$

□

4. We continue the proof of the Proposition 2.3.29. For given (Z_t) we have

$$\left\langle\int_0^tZdM,M\right\rangle_t=\int_0^tZ_{s-}d\left\langle M\right\rangle_s=-\int_0^tZ_s\Psi_{ds}^{''}(0) \ .$$

5. We want to identify

$$-\int_0^tZ_s\Psi_{ds}^{''}(0)=-i\int_0^tV_s(\Psi_{ds}^{'}(u)-\Psi_{ds}^{'}(0)) \ .$$

This naturally leads to

$$Z_s=i\frac{d(\Psi_s^{'}(u)-\Psi_s^{'}(0))}{d\Psi_s^{''}(0)}V_{s-} \ . \tag{3.26}$$

6. Assumption 3 implies that $\mathbb{E}(\int_0^T |Z_s(u)|^2 ds) < \infty$.
7. Since V is an $(\mathcal{F}_t)$-martingale, previous points imply that O is a square integrable $(\mathcal{F}_t)$-martingale.

B) For treating the general case, where μ is a general finite complexe measure, the use of Fubini's theorem is essential. We have to show the following properties.

1. V is a square integrable martingale;
2. $$\int_{\mathbb{R}} d|\mu|(u)\mathbb{E}\left(\int_0^T |Z_s(u)|^2 d\langle M\rangle_s\right) < \infty; \tag{3.27}$$
3. $VM - \int_0^{\cdot} Z_s d\langle M, M\rangle$ is a martingale.
4. $\int_0^T Z_s^2 d\langle M, M\rangle_s < \infty$ so that O is a square integrable $(\mathcal{F}_t)$-martingale.

Point 3. is a consequence of Fubini's, point 2. together with part A) which says that for any $u \in \mathbb{R}$
$(V(u)M - \int_0^{\cdot} \xi_s d\langle M, M\rangle)$ is an $(\mathcal{F}_t)$-martingale. This shows in particular the validity of

$$\langle V, M\rangle_t = \int_0^t Z_s d\langle M, M\rangle_s. \tag{3.28}$$

Point 4. is a consequence of points 2. and 1.
Concerning point 2., we remark that the left-hand side of (3.27) is bounded by

$$\mathbb{E}\left(\int_{\mathbb{R}} d|\mu|(u)\zeta(u)\right) \tag{3.29}$$

where

$$\zeta(u) = \int_0^T \exp(2Re(\Psi_T(u) - \Psi(u)))\left|\frac{d\Psi_t'(u) - \Psi_t'(0)}{d(\Psi_t''(0))}\right|^2 d(-\Psi''(0)).$$

Since $\zeta(u) \leq 2(K(u) + K(0))$ for any $u \in \mathbb{R}$, Assumption 4 2. finally concludes (3.27) and therefore point 2. Point 1. can be proved by similar Fubini's type arguments.

$\square$

Example 2.3.32. *We take $X = M = W$ the classical Wiener process. We have $\Psi_s(u) = -\frac{u^2 s}{2}$ so that $\Psi_s'(u) = -us$ and $\Psi_s''(u) = -s$. So $Z_s = iuV_s$. We recall that*

$$V_s = \mathbb{E}[\exp(iuW_T)|\mathcal{F}_s] = \exp(iuW_s)\exp\left(-u^2\frac{T-s}{2}\right) .$$

In particular, $V_0 = \exp(-\frac{u^2 T}{2})$ *and so*

$$\exp(iuW_T) = i\int_0^T u\exp(iuW_s)\exp\left(-u^2\frac{T-s}{2}\right)dW_s + \exp(-\frac{u^2T}{2}).$$

In fact that expression is classical and it can be derived from Clark-Ocone formula.

Explicit Föllmer-Schweizer decomposition

We introduce a quantity which will be useful in the sequel. For $t \in [0,T], u \in \mathbb{R}$ we set

$$\eta(u,t) = \int_0^t \frac{d(\Psi_s'(u) - \Psi_s{}'(0))}{d(\Psi_s''(0))}\Psi_{ds}'(0)\ . \tag{3.30}$$

Remark 2.3.33. *1. η is defined unambiguously since $d\left(\Psi_t'(u) - \Psi_t'(0)\right)$ is absolutely continuous with respect to $d\Psi_t''(0)$.*

2. η is well-defined, because for any $u \in \mathbb{R}$,

$$\eta(u,t) = \int_0^t \frac{d(\Psi_s'(u) - \Psi_s{}'(0))}{d(\Psi_s''(0))}\frac{d(\Psi_s{}'(0))}{d(\Psi_s''(0))}d\Psi_s''(0)$$

is bounded by Cauchy-Schwarz, taking into account Assumption 3 point 2.

We are now able to evaluate the FS decomposition of $H = f(X_T)$ where f is given by (4.28).

We introduce now a supplementary hypothesis.

Assumption 5. *The quantity*

$$\sup_{u\in\mathrm{supp}\mu, t\in[0,T]} (Re(\eta(u,T) - \eta(u,t)) < \infty\ .$$

Theorem 2.3.34. *Under the assumptions of Proposition 2.3.29 and Assumption 5, the FS decomposition of H is the following*

$$H_t = H_0 + \int_0^t \xi_s dX_s + L_t \quad \textit{with} \quad H_T = H \tag{3.31}$$

and

$$H_t = \int_{\mathbb{R}} H(u)_t\mu(du)\ , \quad \xi_t = \int_{\mathbb{R}} \xi(u)_t\mu(du), \tag{3.32}$$

where

$$\begin{aligned}\xi(u)_t &= i\frac{d(\Psi_t'(u)-\Psi_t'(0))}{d\Psi_t''(0)}H(u)_{t-}\ ,\\ H(u)_t &= \exp\left\{\eta(u,T)-\eta(u,t)+\Psi_T(u)-\Psi_t(u)\right\}e^{iuX_t}\ .\end{aligned} \tag{3.33}$$

Proof. Using Fubini's theorem, with the help of Assumption 5, we reduce the problem to show that

$$H(u)_t = H(u)_0 + \int_0^t \xi(u)_s dX_s + L(u)_t \quad \text{with} \quad H(u)_T = \exp(iuX_T)\ ,$$

for fixed $u \in \mathbb{R}$ where $L(u)$ is a square integrable martingale and $\langle L(u), M\rangle = 0$, where M is the martingale part of the special semimartingale X. Notice that by equation (3.33),

$$H(u)_t = e^{\int_t^T \eta(u,ds)}V(u)_t \quad \text{with} \quad V(u)_t = \exp(iuX_t + \Psi_T(u) - \Psi_t(u))\ .$$

Integrating by parts, gives

$$H(u)_t = H(u)_0 - \int_0^t e^{\int_r^T \eta(u,ds)}V(u)_r\eta(u,dr) + \int_0^t e^{\int_r^T \eta(u,ds)}dV(u)_r\ .$$

We denote again by $Z(u)$ the expression provided by (3.26). We recall that

$$dV(u)_r = Z(u)_r dM_r + dO(u)_r = Z(u)_r(dX_r - dA_r) + dO(u)_r\ ,$$

where A is given by (3.21) and O is a square integrable martingale strongly orthogonal to M (i.e. $\langle M, O\rangle_. = 0$).

$$\begin{aligned}H(u)_t &= H(u)_0 + L(u)_t + \int_0^t e^{\int_r^T \eta(u,ds)}Z(u)_r dX_r - \int_0^t e^{\int_r^T \eta(u,ds)}Z(u)_r(-i\Psi_{dr}'(0))\\ &\quad - \int_0^t e^{\int_r^T \eta(u,ds)}V(u)_r\eta(u,dr)\end{aligned}$$

where

$$L(u)_t = \int_0^t e^{\int_r^T \eta(u,ds)}dO(u)_r\ ,$$

is a martingale strongly orthogonal to M. To conclude, we need to choose η so that

$$\int_0^t Z(u)_r e^{\int_r^T \eta(u,ds)}(-i\Psi_{dr}'(0)) = \int_0^t e^{\int_r^T \eta(u,ds)}V(u)_r\eta(u,dr))\ .$$

This requires

$$\eta(u,dr) = \frac{d(\Psi_r'(u) - \Psi_r{}'(0))}{d(\Psi_r''(0))}\Psi_{dr}'(0)\ .$$

So we define η as in (3.30). $\square$

The Lévy case

Let X be a square integrable Lévy process, with characteristic function $\exp(\Psi(u)t)$. In particular, Ψ is of class $C^2(\mathbb{R})$. We have

$$\frac{d\Psi_t'(u)}{d\Psi_t''(0)} = \frac{\Psi'(u)}{\Psi''(0)} \quad \text{and} \quad \eta(u,t) = t\frac{\Psi'(u) - \Psi'(0)}{\Psi''(0)}\Psi'(0) \ .$$

We remark that Assumptions 2 is verified. Concerning Assumption 3, point 1. is trivial; point 2. is verified because

$$K(u) = \frac{|\Psi'(u)|^2}{-\Psi''(0)} \int_0^T \exp(2(T-t)Re\Psi(u))dt < \infty. \tag{3.34}$$

On the other hand Assumption 5 is verified if

$$\sup_u Re\left(\frac{\Psi'(u)\Psi'(0)}{\Psi''(0)}\right) < \infty \ . \tag{3.35}$$

Since $\Psi'(0) = i\mathbb{E}[X_1]$ and $\Psi''(0) < 0$, (3.35) is fulfilled if

$$\inf_u \mathbb{E}[X_1] Im(\Psi'(u)) > -\infty \ . \tag{3.36}$$

Concerning Assumption 4, (3.34) gives

$$\begin{aligned} K(u) &= \frac{\left|\Psi'(u)\right|^2}{-\Psi''(0)} \int_0^T e^{(T-t)Re\Psi(u)} dt \\ &= \frac{1}{-\Psi''(0)} \frac{|\Psi'(u)|^2}{-Re\Psi(u)} \exp(2Re\Psi(u)T) \end{aligned} \tag{3.37}$$

Example 2.3.35. *We start with the toy model $X_t = \sigma W_t + mt$, $\sigma, m \in \mathbb{R}$. We have $\Psi(u) = -\frac{u^2}{2}\sigma^2 + imu$ so $\Psi'(u) = -u\sigma^2 + im$ and $Im(\Psi'(u)) = m$. Condition 4 is always verified since $K(u) \le \frac{1}{\sigma^2}$ and μ is finite. Condition* (3.36) *is always verified and Assumption 4 is always verified since $K(u) \le -\frac{1}{\sigma^2}$ and μ is finite.*

Remark 2.3.36. *In the examples introduced in Remark 2.3.21, we can show that $u \mapsto \left|\Psi'(u)\right|$ is bounded and so* (3.36) *is fulfilled. Assumption 4 is again satisfied because* (3.37) *implies that $K(u) \le \text{const} \sup |\Psi'(u)$. We recall in fact the following.*

1. *Poisson case*

   We have $\Psi'(u) = i\lambda e^{iu}$.

2. *NIG case*
We have $\Psi'(u) = i\mu + i\delta\,(\beta + iu)\,(\alpha^2 - (\beta + iu)^2)^{-\frac{1}{2}}$ *. Now*

$$|\Psi'(u)| \leq 2\left(|\mu|^2 + 2\delta\sqrt{\frac{\beta^2 + u^2}{(\alpha^2 - \beta^2 + u^2)^2 + 4u^2\beta^2}}\right) .$$

Since $|\alpha| > |\beta|$, $u \mapsto |\Psi'(u)|$ *is bounded.*

3. *Variance Gamma case*
We have $\Psi'(u) = i\mu - \frac{u - i\beta}{\alpha - iu\beta + \frac{u^2}{2}}$ *Clearly* $|\Psi'(u)|$ *is again bounded.*

In conclusion, we can apply Theorem 2.3.34 and we obtain

$$\begin{aligned} V(u)_t &= \exp(iuX_t + (T-t)\Psi(u)) , \\ H(u)_t &= \exp\left((T-t)\Psi(u) + \eta(u,T) - \eta(u,t)\right) e^{iuX_t} , \\ \xi(u)_t &= H_t(u) i \frac{\Psi'(u) - \Psi'(0)}{\Psi''(0)} . \end{aligned}$$

2.3.5 Representation of some contingent claims by Fourier transforms

In general, it is not possible to find a Fourier representation, of the form (3.20), for a given payoff function which is not necessarily bounded or integrable. Hence, it can be more convenient to use the bilateral Laplace transform that allows an extended domain of definition including non integrable functions. We refer to [25], [64] and more recently [31] for such characterizations of payoff functions. This will be done in the next section. However, to illustrate the results of this section restricted to payoff functions represented as classical Fourier transforms, we give here one simple example of such representation extracted from [31]. The payoff of a *self quanto put option* with strike K is

$$f(x) = e^x(K - e^x)_+ \quad \text{and} \quad \hat{f}(u) = \int_{\mathbb{R}} e^{iux} f(x)\,dx = \frac{K^{2+iu}}{(1+iu)(2+iu)} .$$

In this case μ admits an integrable density.

2.4 Föllmer-Schweizer decomposition for exponential of PII processes

In this section, we consisder the case of exponential of PII corresponding to geometric models (such as the Black-Scholes model) much more used in finance than arithmetic models (such as the Bachelier model). The aim of this section is to generalize the results of [49] to the case of PII with possibly non stationary increments. Here again, this generalization is motivated by applications to energy derivatives where forward prices show a volatility term structure that requires the use of models with non stationary increments.

2.4.1 A reference variance measure

We come back to the main optimization problem which was formulated in Section 2.2. We assume that the process S is the discounted price of the non-dividend paying stock which is supposed to be of the form,

$$S_t = s_0 \exp(X_t) \ , \quad \text{for all } t \in [0, T] \ ,$$

where s_0 is a strictly positive constant and X is a semimartingale process with independent increments (PII), in the sense of Definition 2.3.1, but not necessarily with stationary increments.

For notational convenience we introduce the set $\frac{D}{2} = \{z \in \mathbb{C} | 2z \in D\}$.

Remark 2.4.1. *We recall that D is convexe. Consequently we have.*

1. *If $y, z \in \frac{D}{2}$, then $y + z \in D$. If $z \in \frac{D}{2}$ then $\bar{z} \in \frac{D}{2}$ and $2Rez \in D$.*
2. *Since $0 \in D$, clearly $\frac{D}{2} \subset D$.*
3. *Under Assumption 6 below, $2 \in D$ and so $\frac{D}{2} + 1 \subset D$.*

Remark 2.4.2. *Let $\gamma \in \mathbb{R}^*$.*

1. *$\mathbb{E}[\exp(\gamma(X_t - X_s))] > 0$, since $X_t - X_s > -\infty$ a.s.*
2. *$\exp(\gamma(X_t - X_s))$ has a strictly positive variance if $(X_t - X_s)$ is non-deterministic.*

We introduce a new function that will be useful in the sequel.

Definition 2.4.3.

- *For any $t \in [0,T]$, if $z, y \in \frac{D}{2}$ we denote*

$$\rho_t(z,y) = \kappa_t(z+y) - \kappa_t(z) - \kappa_t(y) \ . \tag{4.1}$$

- *To shorten notations $\rho_t : \frac{D}{2} \to \mathbb{C}$ will denote the real valued function such that,*

$$\rho_t(z) = \rho_t(z,\bar{z}) = \kappa_t(2Re(z)) - 2Re(\kappa_t(z)) \ . \tag{4.2}$$

Notice that the last equality results from Remark 2.2.3.

An important technical lemma follows below.

Lemma 2.4.4. *Let $z \in \frac{D}{2}$, with $z \neq 0$, then, $t \mapsto \rho_t(z)$ is strictly increasing if and only if X has no deterministic increments.*

Proof. It is enough to show that X has no deterministic increment if and only if for any $0 \leq s < t \leq T$, the following quantity is positive,

$$\rho_t(z) - \rho_s(z) = \left[\kappa_t\big(2Re(z)\big) - \kappa_s\big(2Re(z)\big)\right] - 2Re\big(\kappa_t(z) - \kappa_s(z)\big) \ . \tag{4.3}$$

By Remark 2.3.11, for all $z \in D$, we have

$$\exp[\kappa_t(z) - \kappa_s(z)] = \mathbb{E}[\exp(z\Delta_s^t)] \ , \quad \text{where } \Delta_s^t = X_t - X_s \ .$$

Applying this property and Remark 2.2.3 1., to the exponential of the first term on the right-hand side of (4.3) yields

$$\begin{aligned}\exp\left[\kappa_t\big(2Re(z)\big) - \kappa_s\big(2Re(z)\big)\right] &= \mathbb{E}[\exp(2Re(z)\Delta_s^t)] = \mathbb{E}[\exp((z+\bar{z})\Delta_s^t)] \\ &= \mathbb{E}[\left|\exp(z\Delta_s^t)\right|^2] \ .\end{aligned}$$

Similarly, for the exponential of the second term on the right-hand side difference of (4.3), one gets

$$\exp\left[2Re\big(\kappa_t(z) - \kappa_s(z)\big)\right] = \exp\left[\big(\kappa_t(z) - \kappa_s(z)\big) + \overline{\big(\kappa_t(z) - \kappa_s(z)\big)}\right] = \left|\mathbb{E}[\exp(z\Delta_s^t)]\right|^2 \ .$$

Hence taking the exponential of $\rho_t(z) - \rho_s(z)$ yields

$$\begin{aligned}\exp[\rho_t(z) - \rho_s(z)] - 1 &= \frac{\mathbb{E}[|\exp(z\Delta_s^t)|^2]}{|\mathbb{E}[\exp(z\Delta_s^t)]|^2} - 1 \ , \\ &= \frac{\mathbb{E}[|\Gamma_s^t(z)|^2]}{|\mathbb{E}[\Gamma_s^t(z)]|^2} - 1 \ , \quad \text{where } \Gamma_s^t(z) = \exp(z\Delta_s^t) \ , \\ &= \frac{Var\left[Re\big(\Gamma_s^t(z)\big)\right] + Var\left[Im\big(\Gamma_s^t(z)\big)\right]}{|\mathbb{E}[\Gamma_s^t(z)]|^2} \ . \end{aligned} \tag{4.4}$$

- If X has a deterministic increment $\Delta_s^t = X_t - X_s$, then $\Gamma_s^t(z)$ is again deterministic and (4.4) vanishes and hence $t \to \rho_t(z)$ is not strictly increasing.
- If X has never deterministic increments, then the nominator is never zero, otherwise $Re\big(\Gamma_s^t(z)\big)$, $Im\big(\Gamma_s^t(z)\big)$ and therefore $\Gamma_s^t(z)$ would be deterministic.

□

From now on, we will always suppose the following assumption.

Assumption 6. *1. (X_t) has no deterministic increments.*

2. $2 \in D$.

Remark 2.4.5. *1. In particular for $\gamma \in \frac{D}{2}$, $\gamma \neq 0$, the function $t \mapsto \rho_t(\gamma)$ is strictly increasing.*

2. If $z = 1$, (4.4) equals $\dfrac{Var\big(\exp(\Delta_s^t)\big)}{\big(\mathbb{E}[\exp(\Delta_s^t)]\big)^2}$, which is a mean-variance quantity.

We continue with a simple observation.

Lemma 2.4.6. *Let I be a compact real interval included in D.*

$$\sup_{x \in I} \sup_{t \leq T} \mathbb{E}[S_t^x] < \infty \ .$$

Proof. Let $t \in [0, T]$ and $x \in I$, we have

$$\mathbb{E}[S_t^x] = s_0^x \exp\{\kappa_t(x)\} \leq \max(1, s_0^{\sup I}) \exp(\sup_{t \leq T, x \in I} |\kappa_t(x)|) \ .$$

since κ is continuous. □

We state now a result that will help us to show that $\kappa_{dt}(z)$ is absolutely continuous with respect to $\rho_{dt}(1) = \kappa_{dt}(2) - 2\kappa_{dt}(1)$.

Lemma 2.4.7. *We consider two positive finite non-atomic Borel measures on $E \subset \mathbb{R}^n$, μ and ν. We suppose the following:*

1. $\mu \ll \nu$;

2. $\mu(I) \neq 0$ for every open ball of E.

Then $\frac{d\mu}{d\nu} := h \neq 0$ ν *a.e. In particular* μ *and* ν *are equivalent.*

Proof. We consider the Borel set

$$B = \{x \in E | h(x) = 0\} \ .$$

We want to prove that $\nu(B) = 0$. So we suppose that there exists a constant $c > 0$ such that $\nu(B) = c > 0$ and another constant ϵ such that $0 < \epsilon < c$. Since ν is a Radon measure, there are compact subsets K_ϵ and $K_{\frac{\epsilon}{2}}$ of E such that

$$K_\epsilon \subset K_{\frac{\epsilon}{2}} \subset B \quad \text{and} \quad \nu(B - K_\epsilon) < \epsilon \ , \quad \nu(B - K_{\frac{\epsilon}{2}}) < \frac{\epsilon}{2} \ .$$

Setting $\epsilon = \frac{c}{2}$, we have

$$\nu(K_\epsilon) > \frac{c}{2} \quad \text{and} \quad \nu(K_{\frac{\epsilon}{2}}) > \frac{3c}{4} \ .$$

By Urysohn lemma, there is a continuous function $\varphi : E \to \mathbb{R}$ such that, $0 \leq \varphi \leq 1$ with

$$\varphi = 1 \text{ on } K_\epsilon \quad \text{and} \quad \varphi = 0 \text{ on } K^c_{\frac{\epsilon}{2}} \ .$$

Now

$$\int_E \varphi(x)\nu(dx) \geq \nu(K_\epsilon) > \frac{c}{2} > 0 \ .$$

By continuity of φ there is an open set $O \subset E$ with $\varphi(x) > 0$ for $x \in O$. Clearly $O \subset K_{\frac{\epsilon}{2}} \subset B$; since O is relatively compact, it is a countable union of balls, and so B contains a ball I. The fact that $h = 0$ on I implies $\mu(I) = 0$ and this contradicts Hypothesis 2. of the statement. Hence the result follows.

□

Remark 2.4.8. *From now on, in this section,* $d\rho_t = \rho_{dt}$ *will denote the measure*

$$d\rho_t = \rho_{dt}(1) = d(\kappa_t(2) - 2\kappa_t(1)) \ . \tag{4.5}$$

According to Remark 2.4.5 1., it is a positive measure which is strictly positive on each interval. This measure will play a fundamental role.

Remark 2.4.9. *1. If* $E = [0, T]$*, then point 2. of Lemma 2.4.7 becomes* $\mu(I) \neq 0$ *for every open interval* $I \subset [0, T]$*.*

2. *The result holds for every normal metric locally connected space E, provided ν are Radon measures.*

Proposition 2.4.10. *Under Assumption 6*

$$d(\kappa_t(z)) \ll d\rho_t \ , \quad \text{for all } z \in D \ . \tag{4.6}$$

Proof. We apply Lemma 2.4.7, with $d\mu = d\rho_t$ and $d\nu = da_t$. Indeed, Proposition 2.3.10 implies Condition 1. of Lemma 2.4.7 and Lemma 2.4.4 implies Condition 2. of Lemma 2.4.7. Therefore, da_t is equivalent to $d\rho_t$. □

Remark 2.4.11. *Notice that this result also holds with $d\rho_t(y)$ instead of $d\rho_t = d\rho_t(1)$, for any $y \in \frac{D}{2}$ such that $Re(y) \neq 0$.*

2.4.2 On some semimartingale decompositions and covariations

Proposition 2.4.12. *We suppose the validity of Assumption 6. Let $y, z \in \frac{D}{2}$. Then S^z is a special semimartingale whose canonical decomposition $S_t^z = M(z)_t + A(z)_t$ satisfies*

$$A(z)_t = \int_0^t S_{u-}^z \kappa_{du}(z) \ , \quad \langle M(y), M(z) \rangle_t = \int_0^t S_{u-}^{y+z} \rho_{du}(z, y) \ , \quad M(z)_0 = s_0^z, \tag{4.7}$$

where $d\rho_u(z)$ is defined by equation (4.2). In particular we have the following:

1. $\langle M(z), M \rangle_t = \int_0^t S_{u-}^{z+1} \rho_{du}(z, 1)$
2. $\langle M(z), M(\bar{z}) \rangle_t = \int_0^t S_{u-}^{2Re(z)} \rho_{du}(z)$.

Proof. The case $y = 1$, follows very similarly to the proof of Lemma 3.2 of [49]. The major tools are integration by parts and Remark 2.3.11 which says that $N(z)_t := e^{-\kappa_t(z)} S_t^z$ is a martingale. The general case can be easily adapted.

□

Remark 2.4.13. *Lemma 2.4.6 implies that $\mathbb{E}\left[| \langle M(y), M(z) \rangle | \right] < \infty$ and so $M(z)$ is a square integrable martingale for any $z \in \frac{D}{2}$.*

2.4.3 On the Structure Condition

If we apply Proposition 2.4.12 with $y = z = 1$, we obtain $S = M + A$ where M is a martingale and

$$A_t = \int_0^t S_{u-}\kappa_{du}(1) \ , \tag{4.8}$$

and

$$\langle M, M\rangle_t = \int_0^t S_{u-}^2(\kappa_{du}(2) - 2\kappa_{du}(1)) = \int_0^t S_{u-}^2 \rho_{du} \ . \tag{4.9}$$

At this point, the aim is to exhibit a predictable $\mathbb{R}$-valued process α such that

1. $A_t = \int_0^t \alpha_s d\langle M\rangle_s$;
2. $K_T = \int_0^T \alpha_s^2 d\langle M\rangle_s$ is bounded.

In that case, according Theorem 2.2.19, there will exist a unique FS decomposition for any $H \in \mathcal{L}^2$ and so the minimization problem (2.1) will have a unique solution, by Theorem 2.2.22.

Proposition 2.4.14. *Under Assumption 6, we have*

$$A_t = \int_0^t \alpha_s d\langle M\rangle_s \ , \tag{4.10}$$

where α is given by

$$\alpha_u := \frac{\lambda_u}{S_{u-}} \quad \textit{with} \quad \lambda_u := \frac{d\kappa_u(1)}{d\rho_u} \ , \quad \textit{for all } u \in [0, T]. \tag{4.11}$$

Moreover the MVT process is given by

$$K_t = \int_0^t \left(\frac{d(\kappa_u(1))}{d\rho_u}\right)^2 d\rho_u \ . \tag{4.12}$$

Corollary 2.4.15. *Under Assumption 6, the structure condition (SC) is verified if and only if*

$$K_T = \int_0^T \left(\frac{d(\kappa_u(1))}{d\rho_u}\right)^2 d\rho_u < \infty \ .$$

In particular, (K_t) is deterministic therefore bounded.

Proof of Proposition 2.4.14. By Proposition 2.4.10, $d\kappa_t(1)$ is absolutely continuous with respect to $d\rho_t$. Setting α_u as in (4.11), relation (4.12) follows from Proposition 2.4.12, expressing $K_t = \int_0^t \alpha_u^2 d\langle M\rangle_u$. $\square$

Lemma 2.4.16. *The space Θ is constituted by all predictable processes v such that*

$$\mathbb{E}\left(\int_0^T v_t^2 S_{t-}^2 d\rho_t\right) < \infty \ .$$

Proof. According to Proposition 2.2.13, the fact that K is bounded and S satisfies (SC), then $v \in \Theta$ holds if and only if v is predictable and $\mathbb{E}[\int_0^T v_t^2 d\langle M, M\rangle_t] < \infty$. Since

$$\langle M, M\rangle_t = \int_0^t S_{s-}^2 d\rho_s \ ,$$

the assertion follows. $\square$

2.4.4 Explicit Föllmer-Schweizer decomposition

We denote by $\mathcal{D}$ the set of $z \in D$ such that

$$\int_0^T \left|\frac{d\kappa_u(z)}{d\rho_u}\right|^2 d\rho_u < \infty. \tag{4.13}$$

From now on, we formulate another assumption which will be in force for the whole section.

Assumption 7. $1 \in \mathcal{D}$.

Remark 2.4.17. *1. Because of Proposition 2.4.10, $\frac{d\kappa_t(z)}{d\rho_t}$ exists for every $z \in D$.*

2. Assumption 7 implies that K is uniformly bounded.

The proposition below will constitute an important step for determining the FS decomposition of the contingent claim $H = f(S_T)$ for a significant class of functions f, see Section 2.4.5.

Proposition 2.4.18. *Let $z \in \mathcal{D} \cap \frac{D}{2}$. with $z+1 \in \mathcal{D}$. (In particular $2Re(z) \in D$).*

1. $S_T^z \in \mathcal{L}^2(\Omega, \mathcal{F}_T)$.

2. *We suppose Assumptions 6 and 7 and we define*

$$\gamma(z,t) := \frac{d(\rho_t(z,1))}{d\rho_t}, \ t \in [0,T]. \tag{4.14}$$

$\int_0^T |\gamma(z,t)|^2 \rho_{dt} < \infty$ *and*

$$\eta(z,t) := \kappa_t(z) - \int_0^t \gamma(z,s)\kappa_{ds}(1) = \kappa_t(z) - \int_0^t \gamma(z,s)\frac{d\kappa_s(1)}{d\rho_s}\rho_{ds} \tag{4.15}$$

is well-defined and $\eta(z,\cdot)$ *is absolutely continuous with respect to* ρ_{ds} *and therefore bounded.*

3. *Under the same assumptions* $H(z) = S_T^z$ *admits a FS decomposition* $H(z) = H(z)_0 + \int_0^T \xi(z)_t dS_t + L(z)_T$ *where*

$$H(z)_t := e^{\int_t^T \eta(z,ds)} S_t^z \ , \tag{4.16}$$

$$\xi(z)_t := \gamma(z,t) e^{\int_t^T \eta(z,ds)} S_{t-}^{z-1} \ , \tag{4.17}$$

$$L(z)_t := H(z)_t - H(z)_0 - \int_0^t \xi(z)_u dS_u \ . \tag{4.18}$$

Proof. 1. is a consequence of Lemma 2.4.6.

2. $\gamma(z,\cdot)$ is square integrable because Assumption 7 and $z, z+1 \in \mathcal{D}$. Moreover η is well-defined since

$$\left(\int_0^T |\gamma(z,s)| \left|\frac{d\kappa_s(1)}{d\rho_s}\right| \rho_{ds}\right)^2 \leq \int_0^T |\gamma(z,s)|^2 \rho_{ds} \int_0^T |\frac{d\kappa_s(1)}{d\rho_s}|^2 \rho_{ds}. \tag{4.19}$$

3. In order to prove that (4.16),(4.17) and (4.18) constitute the FS decomposition of $H(z)$, taking into account Remark 2.2.16 we need to show that

   (a) $H(z)_0$ is $\mathcal{F}_0$-measurable,

   (b) $\langle L(z), M\rangle = 0$,

   (c) $\xi(z) \in \Theta$,

   (d) $L(z)$ is a square integrable martingale.

We proceed similarly to the proof of Lemma 3.3 of [49]. Point (a) is obvious. Partial integration and point 1 of Proposition 2.4.12 yield

$$H(z)_t = H(z)_0+\int_0^t e^{\int_u^T \eta(z,ds)} dM(z)_u-\int_0^t e^{\int_u^T \eta(z,ds)} S_u^z \eta(z,du)+\int_0^t e^{\int_u^T \eta(z,ds)} S_{u-}^z \kappa_{du}(z) \,. \tag{4.20}$$

On the other hand

$$\int_0^t \xi(z)_u dS_u = \int_0^t \xi(z)_u dM_u + \int_0^t \gamma(z,u) e^{\int_u^T \eta(z,ds)} S_{u-}^z \kappa_{du}(1) \,. \tag{4.21}$$

Hence, using expressions (4.20) and (4.21), by definition of η in (4.15), which says $\eta(z,du) = \kappa_{du}(z) - \gamma(z,u)\kappa_{du}(1)$, we obtain

$$L(z)_t = H(z)_t - H(z)_0 - \int_0^t \xi(z)_u dS_u = \int_0^t e^{\int_u^T \eta(z,ds)} dM(z)_u - \int_0^t \xi(z)_u dM_u, \tag{4.22}$$

which implies that $L(z)$ is a local martingale.
From point 1. of Proposition 2.4.12, using (4.17), it follows that

$$\langle L(z), M\rangle_t = \int_0^t e^{\int_u^T \eta(z,ds)} S_{u-}^{z+1} [\rho_{du}(z,1) - \gamma(z,u)\rho_{du}].$$

Then by definition of γ in (4.14), $\rho_{dt}(z,1) = \gamma(z,t)\rho_{dt}$, yields,

$$\langle L(z), M\rangle_t = 0 \,. \tag{4.23}$$

Consequently, point (b) follows. To continue the proof of this proposition we need the lemma below.

Lemma 2.4.19. *For all $z \in \mathbb{C}$ as in Proposition 2.4.18, $d\rho_t$ a.e. we have*

1. $\overline{\gamma(z,t)} = \gamma(\bar{z},t)$;
2. $\overline{\eta(z,t)} = \eta(\bar{z},t)$.

Proof. Using Remark 2.2.3 1) we observe $\bar{z}, \bar{z}+1 \in \mathcal{D}$.

1. By definition of γ in (4.14), $\gamma(z,t)\rho_{dt} = \rho_{dt}(z,1)$. Then, taking the complex conjugate of the integral from 0 to t and using Remark 2.2.3.1 yields,

$$\overline{\int_0^t \gamma(z,s)\rho_{ds}} = \int_0^t \gamma(\bar{z},s)\rho_{ds} \,.$$

2. It is a consequence of the definition of η in (4.15) and point 1.

□

We continue with the proof of point 3. of Proposition 2.4.18. It remains to prove point (d) i.e. that $L(z)$ is a square-integrable martingale for all $z \in D$ and that $Re(\xi(z))$ and $Im(\xi(z))$ are in Θ. (4.22) says that

$$L(z)_t = \int_0^t e^{\int_s^T \eta(z,du)} dM_s(z) - \int_0^t \xi(z)_s dM_s \ .$$

By Proposition 2.4.12, Lemma 2.4.19 and (4.22), it follows

$$\begin{aligned} \left\langle L(z), \overline{L(z)} \right\rangle_t &= \langle L(z), L(\bar{z}) \rangle_t = \left\langle L(z), \int_0^\cdot e^{\int_s^T \eta(\bar{z},du)} dM_s(\bar{z}) \right\rangle_t \\ &= \int_0^t e^{\int_s^T \eta(z,du)} e^{\int_s^T \eta(\bar{z},du)} S_{s-}^{2Re(z)} \rho_{ds}(z) - \int_0^t \xi(z)_s e^{\int_s^T \eta(\bar{z},du)} S_{s-}^{1+\bar{z}} \rho_{ds}(\bar{z},1) \ . \end{aligned} \tag{4.24}$$

Consequently

$$\left\langle L(z), \overline{L(z)} \right\rangle_t = \int_0^t e^{\int_s^T 2Re(\eta(z,du))} S_{s-}^{2Re(z)} [\rho_{ds}(z) - |\gamma(z,s)|^2 \rho_{ds}] \ . \tag{4.25}$$

Taking the expectation in (4.25), using point 2., (4.14), (4.15) and Lemma 2.4.6, we obtain

$$\mathbb{E}\left[\left\langle L(z), \overline{L(z)} \right\rangle_T \right] < \infty \ . \tag{4.26}$$

Therefore, L is a square-integrable martingale.
It remains to prove point (c) i.e. that $\xi(z) \in \Theta$. In view of applying Lemma 2.4.16, we evaluate

$$\int_0^T |\xi(z)_s|^2 S_{s-}^2 \rho_{ds} = \int_0^T |\gamma(z,s)|^2 e^{\int_t^T 2Re(\eta(z,du))} S_{s-}^{2Re(z)} \rho_{ds}. \tag{4.27}$$

Similarly as for (4.25), we can show that the expectation of the left-hand side of (4.27) is finite. This concludes the proof of Proposition 2.4.18. □

2.4.5 FS decomposition of special contingent claims

Now, we will proceed to the FS decomposition of more general contingent claims. We consider now options of the type

$$H = f(S_T) \quad \text{with} \quad f(s) = \int_{\mathbb{C}} s^z \Pi(dz) \ , \tag{4.28}$$

where Π is a (finite) complex measure in the sense of Rudin [68], Section 6.1. An integral representation of some basic European calls can be found later.
We need now the new following assumption.

Assumption 8. *Let* $I_0 = \mathrm{supp}\Pi \cap \mathbb{R}$. *We denote* $I = 2I_0 \cup \{1\}$.

1. I_0 *is compact.*

2. $\forall z \in \mathrm{supp}\Pi, \quad z, z+1 \in \mathcal{D}$.

3. $I_0 \subset \frac{D}{2}$.

4. $\sup_{x\in I} \left\| \frac{d(\kappa_t(x))}{d\rho_t} \right\|_\infty < \infty$.

Remark 2.4.20. 1. *Point 3. of Assumption 8 implies* $\sup_{z\in I+i\mathbb{R}} \|\kappa_{dt}(Re(z))\|_T < \infty$.

2. *Under Assumption 8,* $H = f(S_T)$ *is square integrable. In particular it admits an FS decomposition.*

3. *Because of (4.6) in Proposition 2.4.10, the Radon-Nykodim derivative at Point 4. of Assumption 8, always exists.*

We need now to obtain upper bounds on z for the quantity (4.26). We will first need the following lemma which constitutes the generalization of of Lemma 3.4 of [49] which was stated when X is a Lévy processe. The fact that X does not have stationary increments, constitutes a significant obstacle.

Lemma 2.4.21. *There are positive constants* c_1, c_2, c_3 *such that* $d\rho_s$ *a.e.*

1.
$$\sup_{z\in I_0+i\mathbb{R}} \frac{dRe(\eta(z,s))}{d\rho_s} \leq c_1.$$

2. *For any* $z \in I_0 + i\mathbb{R}$
$$|\gamma(z,s)|^2 \leq \frac{d\rho_s(z)}{d\rho_s} \leq c_2 - c_3 \frac{dRe(\eta(z,s))}{d\rho_s}$$

3.
$$-\sup_{z\in I_0+i\mathbb{R}} \int_0^T 2Re(\eta(z,dt)) \exp(\int_t^T Re(\eta(z,ds))) < \infty.$$

Remark 2.4.22. *1. According to Proposition 2.4.18, $t \mapsto Re(\eta(z,t))$ is absolutely continuous with respect to $d\rho_t$.*

2. We recall that $\mathrm{supp}\Pi$ *is included in* $I_0 + i\mathbb{R}$.

Proof (**of Lemma** 2.4.21). According to Point 3. of Assumption 8 we denote

$$c_{11} := \sup_{x\in I} \left\| \frac{d(\kappa_t(x))}{d\rho_t} \right\|_{\infty} . \tag{4.29}$$

For $z \in I_0 + i\mathbb{R}, t \in [0,T]$, we have

$$\eta(z,t) = \kappa_t(z) - \int_0^t \gamma(z,s)d\kappa_s(1) \quad \text{and} \quad \eta(\bar{z},t) = \kappa_t(\bar{z}) - \int_0^t \gamma(\bar{z},s)d\kappa_s(1).$$

Then, we get $Re(\eta(z,t)) = Re(\kappa_t(z)) - \int_0^t Re(\gamma(z,s))d\kappa_s(1)$. We obtain

$$\begin{aligned} \int_t^T Re(\eta(z,ds)) &\leq Re\left(\kappa_T(z) - \kappa_t(z)\right) + \left| \int_t^T \gamma(z,s)d\kappa_s(1) \right| \\ &= \int_t^T \frac{Re(d\kappa_s(z))}{d\rho_s} d\rho_s + \left| \int_t^T \gamma(z,s)d\kappa_s(1) \right| . \end{aligned} \tag{4.30}$$

Since $\left\langle L(z), \overline{L(z)} \right\rangle_t$ is increasing, and taking into account (4.25), the measure,

$$\left(d\rho_s(z) - |\gamma(z,s)|^2 d\rho_s\right)$$

is non-negative. It follows that

$$\frac{d\rho_s(z)}{d\rho_s} - |\gamma(z,s)|^2 \geq 0 \ , \quad d\rho_s \ a.e. \tag{4.31}$$

Remark 2.4.23. *By* (4.31)*, in particular the density* $\frac{d\rho_s(z)}{d\rho_s}$ *is non-negative* $d\rho_s$ *a.e.*

Consequently,

$$2\frac{dRe(\kappa_s(z))}{d\rho_s} \leq \frac{d\kappa_s(2Re(z))}{d\rho_s} \ , \quad d\rho_s \ a.e. \tag{4.32}$$

In order to prove 1. it is enough to verify that, for some $c_0 > 0$,

$$\frac{dRe(\eta(z,s))}{d\rho_s} \leq c_0 + \frac{1}{2}\frac{dRe(\kappa_s(z))}{d\rho_s} \quad d\rho_s \ a.e. \tag{4.33}$$

In fact, (4.32) and Assumption 8 point 3. and (4.29), imply that

$$\frac{dRe(\eta(z,s))}{d\rho_s} \leq c_0 + \frac{1}{2}c_{11} =: c_1. \tag{4.34}$$

To prove (4.33) it is enough to show that

$$Re(\eta(z,T)-\eta(z,t)) \leq c_0(\rho_T-\rho_t) + \frac{1}{2}Re(\kappa_T(z)-\kappa_t(z)), \quad \forall t \in [0,T]. \tag{4.35}$$

Again Assumption 8 point 3. implies that

$$\left|\int_t^T \gamma(z,s)d\kappa_s(1)\right| \leq c_{12}\int_t^T |\gamma(z,s)|d\rho_s \ , \tag{4.36}$$

where $c_{12} = \|\frac{d\kappa_s(1)}{d\rho_s}\|_\infty$. Using (4.31), and Assumption 8 it follows

$$|\gamma(z,s)|^2 \leq \frac{d\rho_s(z)}{d\rho_s} = \frac{d\kappa(2Re(z))}{d\rho_s} - \frac{2dRe(\kappa_s(z))}{d\rho_s} \leq c_{11} - \frac{2dRe(\kappa_s(z))}{d\rho_s}. \tag{4.37}$$

This implies that

$$c_{12}^2\,|\gamma(z,s)|^2 \leq \left(c_{13}^2 + \frac{1}{4}\left(\frac{dRe(\kappa_s(z))}{d\rho_s}\right)^2\right) \ ,$$

where $c_{13} > 0$ is chosen such that $c_{13}^2 \geq 4c_{12}^4 + c_{12}^2 c_{11}$. Consequently

$$\left|\int_t^T \gamma(z,s)d\kappa_s(1)\right| \leq \int_t^T d\rho_s\left(c_{13} + \frac{1}{2}\left|\frac{dRe(\kappa_s(z))}{d\rho_s}\right|\right) \ .$$

Coming back to (4.30), we obtain

$$\begin{aligned} Re(\eta(z,T)-\eta(z,t)) &\leq \int_t^T \left(\frac{Re(d\kappa_s(z))}{d\rho_s} + c_{13} + \frac{1}{2}\left|\frac{Re(d\kappa_s(z))}{d\rho_s}\right|\right)d\rho_s \\ &\leq \int_t^T \left(\frac{1}{2}\frac{Re(d\kappa_s(z))}{d\rho_s} + \left(\frac{Re(d\kappa_s(z))}{d\rho_s}\right)^+ + c_{13}\right)d\rho_s \end{aligned}$$

(4.32) and Assumption 8 allow to establish

$$Re(\eta(z,T)-\eta(z,t)) \leq \int_t^T d\rho_s\left(c_0 + \frac{1}{2}\frac{dRe(\kappa_s(z))}{d\rho_s}\right) \ , \tag{4.38}$$

where $c_0 = \frac{c_{11}}{2} + c_{13}$. This concludes the proof of point 1.
In order to prove point 2. we first observe that (4.33) implies

$$-\frac{dRe(\kappa_s(z))}{d\rho_s} \leq 2\left(c_0 - \frac{dRe(\eta(z,s))}{d\rho_s}\right) \tag{4.39}$$

$d\rho_s$ a.e. (4.37) implies

$$|\gamma(z,s)|^2 \leq c_{21} - 4\frac{dRe(\eta(z,s))}{d\rho_s} \ , \tag{4.40}$$

where $c_{21} = c_{11} + 4c_0$. Point 2. is now established with $c_2 = c_{21}$ and $c_3 = 4$.
We continue with the proof of point 3. We decompose

$$Re(\eta(z,t)) = A^+(z,t) - A^-(z,t) \ ,$$

where

$$A^+(z,t) = \int_0^t \left(\frac{dRe(\eta(z,s))}{d\rho_s}\right)_+ d\rho_s \ , \quad \text{and} \quad A^-(z,t) = \int_0^t \left(\frac{dRe(\eta(z,s))}{d\rho_s}\right)_- d\rho_s \ .$$

$A^+(z,.)$ and $A^-(z,.)$ are increasing non negative functions. Moreover point 1. implies

$$A^+(z,t) \leq c_1\rho_t \ .$$

At this point, for $z \in I_0 + i\mathbb{R}$

$$\begin{aligned}
-\int_0^T Re(\eta(z,dt))e^{\int_t^T 2Re(\eta(z,ds))} &= \int_0^T \big(A^-(z,dt) - A^+(z,dt)\big)\, e^{2\int_t^T Re(\eta(z,ds))} \\
&\leq \int_0^T A^-(z,dt) e^{2\left(A^+(z,T)-A^+(z,t)\right)} e^{-2\left(A^-(z,T)-A^-(z,t)\right)} \\
&\leq e^{2c_1\rho_T} \int_0^T e^{-2\left(A^-(z,T)-A^-(z,t)\right)} A^-(z,dt) \\
&= \frac{e^{2c_1\rho_T}}{2}\left\{1 - e^{-2A^-(z,T)}\right\} \leq \frac{e^{2c_1\rho_T}}{2} \ ,
\end{aligned}$$

which concludes the proof of point 3 of Lemma 2.4.21.

□

By Lemma 2.4.6, it follows

$$c_4 := \sup_{x\in I, s\leq T} \mathbb{E}\left[S_s^x\right] < \infty \ . \tag{4.41}$$

Theorem 2.4.24. *Let Π be a finite complex-valued Borel measure on $\mathbb{C}$.
Suppose Assumptions 6, 7, 8. Any complex-valued contingent claim $H = f(S_T)$, where f is of the form (4.28), and $H \in \mathcal{L}^2$, admits a unique FS decomposition $H = H_0 + \int_0^T \xi_t dS_t + L_T$ with the following properties.*

1. $H \in \mathcal{L}^2$ and

$$H_t = \int H(z)_t \Pi(dz), \quad \xi_t = \int \xi(z)_t \Pi(dz), \quad L_t = \int L(z)_t \Pi(dz),$$

where for $z \in \mathrm{supp}(\Pi)$, $H(z), \xi(z)$ and $L(z)$ are the same as those introduced in Proposition 2.4.18 and we convene that they vanish if $z \notin \mathrm{supp}(\Pi)$.

2. Previous decomposition is real-valued if f is real-valued.

Remark 2.4.25. *Taking $\Pi = \delta_{z_0}(dz)$, $z_0 \in \mathbb{C}$, Assumption 8 is equivalent to the assumptions of Proposition 2.4.18.*

Proof. a) $f(S_T) \in \mathcal{L}^2$ since by Jensen,

$$E\left|\int_{\mathbb{C}} \Pi(dz) S_T^z\right|^2 \leq \int_{\mathbb{C}} |\Pi|(dz) E|S_T^{2Rez}||\Pi|(\mathbb{C}) \leq \sup_{x \in I_0} E(S_T^{2x})|\Pi|(\mathbb{C})^2,$$

where $|\Pi|$ denotes the total variation of the finite measure Π. Previous quantity is bounded because of Lemma 2.4.18.

We go on with the FS decomposition. We would like to prove first that H and L are well defined square-integrable processes and $E(\int_0^T |\xi_s|^2 d\langle M\rangle_s) < \infty$.

By Jensen's inequality, we have

$$\mathbb{E}\left|\int_{\mathbb{C}} L(z)_t \Pi(dz)\right|^2] \leq \mathbb{E}\left(\int_{\mathbb{C}} |\Pi|(dz)|L_t(z)|_t^2\right)|\Pi(\mathbb{C})| = \int_{\mathbb{C}} |\Pi|(dz)\mathbb{E}[|L_t(z)|_t^2]|\Pi(\mathbb{C})|.$$

Similar calculations allow to show that

$$\mathbb{E}[\xi_t^2] \leq |\Pi|(\mathbb{C}) \int_{\mathbb{C}} |\Pi|dz)\mathbb{E}[|\xi_t(z)|^2] \quad \text{and} \quad \mathbb{E}[L_t^2] \leq |\Pi(\mathbb{C})| \int_{\mathbb{C}} |\Pi|(dz)\mathbb{E}[|L_t(z)|^2] \; .$$

We will show now that

- (A1): $\sup_{t \leq T, z \in \mathrm{supp}\Pi} \mathbb{E}[|H_t(z)|^2] < \infty$;
- (A2): $\int_{\mathbb{C}} |\Pi|(dz)\mathbb{E}[|L_t(z)|_t^2] < \infty$;
- (A3):
$$E\left(\int_0^T d\rho_t S_t^2 \int_{\mathbb{C}} |\xi_t(z)|^2 |\Pi|(dz)\right) < \infty \, .$$

(A1): Since $H(z)_t = e^{\int_t^T \eta(z,ds)} S_t^z$, we have

$$|H(z)_t|^2 = H(z)_t \overline{H(z)_t} = e^{\int_t^T 2Re(\eta(z,ds))} S_t^{2Re(z)} \ ,$$

so

$$\mathbb{E}[|H(z)_t|^2] = e^{\int_t^T 2Re(\eta(z,ds))} \mathbb{E}[S_t^{2Re(z)}] \leq c_4 e^{\int_t^T 2Re(\eta(z,ds))},$$

where c_4 was defined in (4.41). Lemma 2.4.21 imply (A1). Therefore (H_t) is a well-defined square-integrable process.

(A2): $\mathbb{E}[|L_t(z)|^2] \leq \mathbb{E}[|L_T(z)|^2] = \mathbb{E}[\left\langle L(z), \overline{L(z)} \right\rangle_T]$, where the first inequality is due to the fact that $|L_t(z)|^2$ is a submartingale.

$$\mathbb{E}\left[\left\langle L(z), \overline{L(z)} \right\rangle_T\right] = \mathbb{E}\left[\int_0^T e^{\int_s^T 2Re(\eta(z,du)} S_{s-}^{2Re(z)} \left[d\rho_s(z) - |\gamma(z,s)|^2 d\rho_s\right]\right] \ .$$

By Fubini's, Lemma 2.4.6 and (4.25), we have

$$\begin{aligned}
\mathbb{E}\left[\left\langle L(z), \overline{L(z)} \right\rangle_T\right] &= \int_0^T e^{\int_s^T 2Re(\eta(z,du)} \mathbb{E}[S_{s-}^{2Re(z)}] \left[\frac{d\rho_s(z)}{d\rho_s} - |\gamma(z,s)|^2\right] d\rho_s \\
&\leq c_4 \int_0^T e^{\int_s^T 2Re(\eta(z,du)} \left[\frac{d\rho_s(z)}{d\rho_s}\right] d\rho_s.
\end{aligned}$$

According to Lemma 2.4.21 point 2, previous expression is bounded by $c_4 I(z)$, where

$$\begin{aligned}
I(z) &:= \int_0^T d\rho_t \exp\left(\int_t^T 2Re(\eta(z,ds)) \left[c_2 - c_3 \frac{dRe(\eta(z,t))}{d\rho_t}\right]\right) \\
&= c_2 I_1(z) + c_3 I_2(z) \ ,
\end{aligned} \tag{4.42}$$

where

$$\begin{aligned}
I_1(z) &= \int_0^T d\rho_t \exp\left(\int_t^T 2Re(\eta(z,ds))\right) \\
I_2(z) &= \int_0^T \exp\left(\int_t^T 2Re(\eta(z,ds))\right) Re(\eta(z,ds))
\end{aligned}$$

Using Lemma 2.4.21, we obtain

$$\sup_{z \in I_0 + i\mathbb{R}} |I_1(z)| \leq \rho_T \exp(2c_1 \rho_T) \quad \text{and} \quad \sup_{z \in I_0 + i\mathbb{R}} |I_2(z)| < \infty \ , \tag{4.43}$$

and so

$$\sup_{z\in I_0+i\mathbb{R}} \mathbb{E}\left[\left\langle L(z),\overline{L(z)}\right\rangle_T\right] < \infty \; . \tag{4.44}$$

This concludes (A2).
We verify now the validity of (A3). This requires to control

$$\mathbb{E}\left[\int_0^T \rho_{dt} S_t^2 \left(\int_{\mathbb{C}} |\Pi|(dz)|\xi(z)_t|^2\right)\right] \leq \mathbb{E}\left[\int_0^T \rho_{dt} S_t^2 \left(\int_{\mathbb{C}} |\Pi|(dz)\left|\gamma(z,t)\exp\left(\int_t^T Re(\eta(z,ds))\right) S_t^{z-1}\right|^2\right)\right]$$

Using Jensen's inequality, this is smaller or equal than

$$|\Pi(\mathbb{C})| \int_{\mathbb{C}} |\Pi|(dz) \int_0^T \rho_{dt}\mathbb{E}\left[S_t^{2Re(z)}\right] |\gamma(z,t)|^2 \exp\left(2\int_t^T Re(\eta(z,ds))\right) \; .$$

Lemma 2.4.21 gives the upper bound

$$c_4|\Pi|(\mathbb{C}) \int_{\mathbb{C}} |\Pi|(dz) I(z) \; ,$$

where $I(z)$ was defined in (4.43). Since Π is finite and because of (4.44), (A3) is now established.
In order to conclude, it remains to show that L is an $(\mathcal{F}_t)$-martingale which is strongly orthogonal to M. This can be established similarly as in [49], Proposition 3.1, by making use of Fubini's theorem and Fubini's theorem for stochastic integrals (cf. [63], Theorem IV.46) and (A1), (A2), (A3).

Consequently, (H_0, ξ, L) provide a (possibly complexe) FS decomposition of H.

b) It remains to prove that the decomposition is real-valued. Let (H_0, ξ, L) and $(\overline{H_0}, \overline{\xi}, \overline{L})$ be two FS decomposition of H. Consequently, since H and (S_t) are real-valued, we have

$$0 = H - \overline{H} = (H_0 - \overline{H}_0) + \int_0^T (\xi_s - \overline{\xi}_s) dS_s + (L_T - \overline{L}_T) \; ,$$

which implies that $0 = Im(H_0) + \int_0^T Im(\xi_s)dS_s + Im(L_T)$. By Theorem 2.2.19, the uniqueness of the real-valued Föllmer-Schweizer decomposition yields that the processes (H_t),(ξ_t) and (L_t) are real-valued.

□

2.4.6 Representation of some typical contingent claims

We used some integral representations of payoffs of the form (4.28). We refer to [25], [64] and more recently [31], for some characterizations of classes of functions which admit this kind of representation. In order to apply the results of this paper, we need explicit formulae for the complex measure Π in some example of contingent claims.

Call

The first example is the European Call option $H = (S_T - K)_+$. We have two representations of the form (4.28) which result from the following lemma.

Lemma 2.4.26. *Let $K > 0$, the European Call option $H = (S_T - K)_+$ has two representations of the form (4.28):*

1. *For arbitrary $R > 1$, $s > 0$, we have*

$$(s - K)_+ = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} s^z \frac{K^{1-z}}{z(z-1)} dz \ . \tag{4.45}$$

2. *For arbitrary $0 < R < 1$, $s > 0$, we have*

$$(s - K)_+ - s = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} s^z \frac{K^{1-z}}{z(z-1)} dz \ . \tag{4.46}$$

Put

Lemma 2.4.27. *Let $K > 0$, the European Put option $H = (K - S_T)_+$ gives for an arbitrary $R < 0$, $s > 0$*

$$(K - s)_+ = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} s^z \frac{K^{1-z}}{z(z-1)} dz \ . \tag{4.47}$$

2.5 The solution to the minimization problem

2.5.1 Mean-Variance Hedging

FS decomposition will help to provide the solution to the global minimization problem. Next theorem deals with the case where the underlying process is a PII.

Theorem 2.5.1. *Let $X = (X_t)_{t\in[0,T]}$ be a process with independent increments with log-characteristic function Ψ_t. Let $H = f(X_T)$ where f is of the form (3.20). We suppose that the PII, X, satisfies Assumptions 2, 3, 4 and 5. Then, the variance-optimal capital V_0 and the variance-optimal hedging strategy φ, solution of the minimization problem* (2.1)*, are given by*

$$V_0 = H_0 \ , \tag{5.1}$$

and the implicit expression

$$\varphi_t = \xi_t + \alpha_t(H_{t-} - V_0 - \int_0^t \varphi_s dS_s) \ , \tag{5.2}$$

where the processes (H_t),(ξ_t) and (λ_t) are defined by

$$H_t = \int_{\mathbb{R}} H(u)_t \mu(du) \ , \quad \xi_t = \int_{\mathbb{R}} i\frac{d(\Psi_t^{'}(u) - \Psi_t^{'}(0))}{d\Psi_t^{''}(0)} H(u)_t \mu(du) \quad \text{and} \quad \alpha_t = i\frac{d\Psi_t^{'}(0)}{d\Psi_t^{''}(0)} \ , \tag{5.3}$$

and

$$H(u)_t = e^{\eta(u,T)-\eta(u,t)+\Psi_T(u)-\Psi_t(u)}\, e^{iuX_{t-}} \quad \text{with} \quad \eta(u,t) = i\int_0^t \frac{d\Psi_t^{'}(0)}{d\Psi_t^{''}(0)} d\left(\Psi_s^{'}(u) - \Psi_s^{'}(0)\right) \tag{5.4}$$

The optimal initial capital is unique. The optimal hedging strategy $\varphi_t(\omega)$ is unique up to some $(P(d\omega) \otimes dt)$-null set.

Proof. Since K is deterministic, the optimality follows from Theorem 2.3.34, Theorem 2.2.22 and Corollary 2.2.24. Uniqueness follows from Theorem 2.2.21. We recall that α was given in (3.15). □

Next theorem deals with the case where the payoff to hedge is given as a bilateral Laplace transform of the exponential of a PII. It is an extension of Theorem 3.3 of [49] to PII with no stationary increments.

Theorem 2.5.2. *Let $X = (X_t)_{t\in[0,T]}$ be a process with independent increments with cumulant generating function κ. Let $H = f(e^{X_T})$ where f is of the form (4.28). We assume the validity of Assumptions 6, 7, 8. The variance-optimal capital V_0 and the variance-optimal hedging strategy φ, solution of the minimization problem* (2.1)*, are given by*

$$V_0 = H_0 \tag{5.5}$$

and the implicit expression

$$\varphi_t = \xi_t + \frac{\lambda_t}{S_{t-}}(H_{t-} - V_0 - \int_0^t \varphi_s dS_s) \ , \tag{5.6}$$

where the processes (H_t), (ξ_t) and (λ_t) are defined by

$$\gamma(z,t) := \frac{d\rho_t(z,1)}{d\rho_t} \quad \text{with} \quad \rho_t(z,y) = \kappa_t(z+y) - \kappa_t(z) - \kappa_t(y) \ , \tag{5.7}$$

$$\eta(z,dt) := \kappa_{dt}(z) - \gamma(z,t)\kappa_{dt}(1) \ , \tag{5.8}$$

$$\lambda_t := \frac{d(\kappa_t(1))}{d\rho_t} \ , \tag{5.9}$$

$$H_t := \int_{\mathbb{C}} e^{\int_t^T \eta(z,ds)} S_t^z \Pi(dz) \ , \tag{5.10}$$

$$\xi_t := \int_{\mathbb{C}} \gamma(z,t) e^{\int_t^T \eta(z,ds)} S_{t-}^{z-1} \Pi(dz) \ . \tag{5.11}$$

The optimal initial capital is unique. The optimal hedging strategy $\varphi_t(\omega)$ is unique up to some $(P(d\omega) \otimes dt)$-null set.

Remark 2.5.3. *The mean variance tradeoff process can be expressed as follows, see (4.12):*

$$K_t = \int_0^t \frac{d\kappa_u(1)}{d\rho_u} \kappa_{du}(1) \ .$$

Proof of Theorem 2.5.2. Since K is deterministic, the optimality follows from Theorem 2.4.24, Theorem 2.2.22 and Corollary 2.2.24. We recall that α was calculated in (4.11). Uniqueness follows from Theorem 2.2.21.

□

When the underlying price is an exponential of PII process, we evaluate the so called **variance of the hedging error** of the contingent claim H i.e. the quantity $\mathbb{E}[(V_0 + G_T(\varphi) - H)^2]$, where V, φ and H were defined at Theorem 2.5.2.

Theorem 2.5.4. *Under the assumptions of Theorem 2.5.2, the variance of the hedging error equals*

$$J_0 := \left(\int_{\mathbb{C}} \int_{\mathbb{C}} J_0(y,z) \Pi(dy) \Pi(dz) \right) ,$$

where

$$J_0(y,z) := \begin{cases} s_0^{y+z} \int_0^T \beta(y,z,dt) e^{\kappa_t(y+z)+\alpha(y,z,t)} & : \ y,z \in \mathrm{supp}\Pi \\ 0 & : \ \textit{otherwise.} \end{cases}$$

and

$$\begin{aligned} \alpha(y,z,t) &:= \eta(z,T) - \eta(z,t) - (\eta(y,T) - \eta(y,t)) - \int_t^T \left(\frac{d\kappa_s(1)}{d\rho_s} \right)^2 d\rho_s \ , \\ \beta(y,z,t) &:= \rho_t(y,z) - \int_0^t \gamma(z,s) \rho_{ds}(y,1) \ . \end{aligned}$$

Remark 2.5.5. *We have*

$$\alpha(y,z,t) = (\eta(z,T) - \eta(z,t)) - (\eta(y,T) - \eta(y,t)) - (K_T - K_t) \ ,$$

where K is the MVT process.

Proof [of Theorem 2.5.4]. Since $X_0 = 0$, $\mathcal{F}_0$ is the trivial σ-field, therefore $L_0 = 0$, because it is mean-zero and deterministic.

The quadratic error can be calculated using Corollary 2.2.24 and Theorem 2.2.22 3. They give

$$\mathbb{E}\left[\int_0^T \exp\left\{ -(K_T - K_s) \right\} d\left\langle L \right\rangle_s \right] \ , \tag{5.12}$$

where L is the remainder martingale in the FS decomposition of H. We proceed now to the evaluation of $\langle L \rangle$.

Similarly to the proof of Theorem 3.2 pf [49], using (4.24), Remark 2.2.4, the bilinearity of the covariation and (4.44), it is possible to show that

$$\int \int \left\langle L(y), L(z) \right\rangle_t \Pi(dy) \Pi(dz),$$

is a well-defined, continuous, predictable, with bounded variation complex-valued process and

$$\left\langle L, L \right\rangle_t = \int \int \left\langle L(y), L(z) \right\rangle_t \Pi(dy) \Pi(dz). \tag{5.13}$$

It remains to evaluate $\langle L(y), L(z)\rangle$ for $y, z \in \mathrm{supp}(\Pi)$.
We know by Proposition 2.4.12 that for all $y, z \in \frac{D}{2}$,

$$\langle M(y), M(z)\rangle_t = \int_0^t S_{u-}^{y+z} \rho_{du}(y,z) \ .$$

Using the same terminology of Proposition 2.4.18, similarly to (4.25) we have

$$\langle L(y), L(z)\rangle_t = \int_0^t e^{\int_s^T (\eta(z,du)+\eta(y,du))} S_{s-}^{y+z} \left[\rho_{ds}(y,z) - \gamma(z,s)\rho_{ds}(y,1)\right] \ .$$

Hence,

$$\langle L(y), L(z)\rangle_t = \int_0^t e^{\int_s^T (\eta(z,du)+\eta(y,du))} S_{s-}^{y+z} \beta(y,z,ds) \ .$$

We come back to (5.12). Recalling Remark 2.5.5 we have

$$\int_0^T e^{-(K_T-K_t)} d\langle L(y), L(z)\rangle_t = \int_0^T e^{\alpha(y,z,t)} S_{t-}^{y+z} \beta(y,z,dt).$$

Since $\mathbb{E}[S_{t-}^{y+z}] = s_0^{y+z} e^{\kappa_t(y+z)}$, an application of Fubini's theorem yields

$$\mathbb{E}\left(\int_0^T e^{-(K_T-K_t)} d\langle L(y), L(z)\rangle_t\right) = s_0^{y+z} \int_0^T e^{\alpha(y,z,t)+\kappa_t(y+z)} \beta(y,z,dt) \ . \tag{5.14}$$

which equals $J_0(y,z)$. (5.13), (5.14) and Fubini's theorem imply

$$\int_0^T e^{-(K_T-K_t)} d\langle L, L\rangle_t = \int_{\mathbb{C}}\int_{\mathbb{C}}\int_0^T e^{-(K_T-K_t)} d\langle L(y), L(z)\rangle_t \Pi(dy)\Pi(dz) \ ,$$

hence

$$\begin{aligned}\mathbb{E}\left[\int_0^T e^{-(K_T-K_t)} d\langle L, L\rangle_t\right] &= \int_{\mathbb{C}}\int_{\mathbb{C}} \mathbb{E}\left[\int_0^T e^{-(K_T-K_t)} d\langle L(y), L(z)\rangle_t\right] \Pi(dy)\Pi(dz) \ ,\\ &= \int_{\mathbb{C}}\int_{\mathbb{C}} J_0(y,z)\Pi(dy)\Pi(dz).\end{aligned}$$

This concludes the proof of Theorem 2.5.4. □

2.5.2 The exponential Lévy case

In this section, we specify rapidly the results concerning FS decomposition and the minimization problem when (X_t) is a Lévy process (Λ_t). Using the fact that (Λ_t) is a process with independent stationary increments it is not difficult to show that

$$\kappa_t(z) = t\kappa^{\Lambda}(z) \ , \tag{5.15}$$

where $\kappa^\Lambda(z) = \kappa_1(z)$, $\kappa^\Lambda : D \to \mathbb{C}$. Since for every $z \in D$, $t \mapsto \kappa_t(z)$ has bounded variation then $X = \Lambda$ is a semimartingale; moreover Proposition 2.3.16 implies that κ^Λ is continuous.

We make the following hypothesis.

Assumption 9. *1.* $2 \in D$;

2. $\kappa^\Lambda(2) - 2\kappa^\Lambda(1) \neq 0$.

Remark 2.5.6. *1.* $\rho_{dt} = \big(\kappa^\Lambda(2) - 2\kappa^\Lambda(1)\big)\, dt$;

2. $\dfrac{d\kappa_t}{d\rho_t}(z) = \dfrac{1}{\kappa^\Lambda(2) - 2\kappa^\Lambda(1)}\kappa^\Lambda(z)$ *for any* $t \in [0, T], z \in D$ *; so* $D = \mathcal{D}$*.*

3. Assumptions 6, and 7 are verified.

4. Assumption 8 4. is always verified if I_0 *is compact since* κ^Λ *is continuous.*

5. Since $D = \mathcal{D}$*, Assumption 8 2. is verified if Assumption 8 3. is fulfilled.*

Again we denote the process S as

$$S_t = s_0 \exp(X_t) = s_0 \exp(\Lambda_t) \ .$$

It remains to verify points 1. and 3. of Assumption 8 which of course depends on the contingent claim.

Example 2.5.7. *1.* $H = (S_T - K)_+$*. We choose the second representation for the call. So, for* $0 < R < 1$*,*

$$I_0 = supp(\Pi) \cap \mathbb{R} = \{R, 1\} \, .$$

Assumption 8 1. is clearly satisfied. Since $2 \in D$ *by Assumption 9, in this case Assumption 8.3 reduces to* $2R \in D$*. This is always satisfied since* $D \supset [0, 2]$ *and it is convex.*

2. $H = (K - S_T)_+$*. We recall that* $R < 0$ *and so*

$$I_0 = supp(\Pi) \cap \mathbb{R} = \{R\}.$$

Again of course point 1. of Assumption 8 is fulfilled. Point 3. gives again $2R \in D$*. Now* $2R$ *is a negative value but this is not a restriction provided that* D *contains some negative values since we have the degree of freedom for choosing* R*.*

Remark 2.5.8. *We come back to the examples introduced in Remark 2.3.21. In all the three cases, Assumption 9 is verified if $2 \in D$. This happens in the following situations:*

1. *always in the Poisson case;*
2. *if $\Lambda = X$ is a NIG process and if $2 \le \alpha - \beta$;*
3. *if $\Lambda = X$ is a VG process and if $2 < -\beta + \sqrt{\beta^2 + 2\alpha}$.*

Theorem 2.5.2 allows to reobtain the results stated in [49]. They will appear as a particular case of Corollary 2.5.16.

Remark 2.5.9. *If X is a Poisson process with parameter $\lambda > 0$ then the quadratic error is zero. In fact, the quantities*

$$\begin{aligned}
\kappa^\Lambda(z) &= \lambda(\exp(z) - 1)) \\
\rho_t(y, z) &= \lambda t(\exp(y) - 1)(\exp(z) - 1) \\
\gamma(z, t) &= \frac{\kappa^\Lambda(z+1) - \kappa^\Lambda(z) - \kappa^\Lambda(1)}{\kappa^\Lambda(2) - 2\kappa^\Lambda(1)} t = \frac{\exp(z) - 1}{e - 1}
\end{aligned}$$

imply that $\beta(y, z, t) = 0$ for every $y, z \in \mathbb{C}, t \in [0, T]$.

Therefore $J_0(y, z, t) \equiv 0$. In particular all the options of type (4.28) are perfectly hedgeable.

2.5.3 Exponential of a Wiener integral driven by a Lévy process

Let Λ be a Lévy process. The cumulant function of Λ_t equals $\kappa_t^\Lambda(z) = t\kappa_1^\Lambda(z)$ for $\kappa_1^\Lambda = \kappa^\Lambda : D_\Lambda \to \mathbb{C}$. We formulate the following hypothesis:

Assumption 10. 1. *There is $r > 0$ such that $r \in D_\Lambda$.*

2. *$\kappa^\Lambda(2) - 2\kappa^\Lambda(1) \neq 0$.*
3. *Let $\varepsilon > 0$ such that $2\varepsilon \le r$ and $l : [0, T] \to [\varepsilon, r/2]$ be a (deterministic continuous) function.*

We consider the PII process $X_t = \int_0^t l_s d\Lambda_s$.

Remark 2.5.10. *According to Lemma 2.4.4 for every $\gamma > 0$, such that $\gamma \in D$,*

$$\kappa^\Lambda(2\gamma) - 2\kappa^\Lambda(\gamma) > 0 \ . \tag{5.16}$$

Remark 2.5.11. *1. Lemma 2.3.24 says that D contains $D_{\varepsilon,r} := \left\{x \in \mathbb{R} \,|\, \varepsilon x, \frac{rx}{2} \in D_\Lambda\right\} + i\mathbb{R}$, and $\kappa_t(z) = \int_0^t \kappa^\Lambda(zl_s)ds$.*

2. $\rho_t = \int_0^t \left(\kappa^\Lambda(2l_s) - 2\kappa^\Lambda(l_s)\right) ds$;

3. $2 \in D$; X is a PII semimartingale since $t \mapsto \kappa_t(2)$ has bounded variation, see Lemma 2.3.14.

4. $1 \in D_{\varepsilon,r}$ since $0, r \in D_\Lambda$.

Remark 2.5.12. *If $l \equiv 1$ then $X = \Lambda$ and the validity of Assumption 10 is equivalent to the validity of Assumption 9. In fact if Assumption 10 is verified then, setting $r = 2, \varepsilon = 1$, Assumption 9 is verified. The converse is a consequence of Remark 2.5.11 3.*

Proposition 2.5.13. *Assumptions 6 and 7 are verified. Moreover $D_{\varepsilon,r} \subset \mathcal{D}$.*

Proof. 1. Using Lemma 2.4.4, Assumption 6 is verified if we show that $t \mapsto \rho_t(1) = \kappa_t(2) - 2\kappa_t(1)$ is strictly increasing. Now

$$\kappa_t(2) - 2\kappa_t(1) = \int_0^t \left(\kappa^\Lambda(2l_s) - 2\kappa^\Lambda(l_s)\right) ds \ .$$

Inequality (5.16) and Lemma 2.4.4 imply that $\forall s \in [0, T]$

$$\kappa^\Lambda(2l_s) - 2\kappa^\Lambda(l_s) > 0 \ .$$

In fact, Λ has no deterministic increments. This shows Assumption 6.

2. For $z \in D_{\varepsilon,r}$, by Remark 2.5.11 1. we have

$$\left|\frac{d\kappa_t(z)}{d\rho_t}\right| = \left|\frac{\kappa^\Lambda(zl_t)}{\kappa^\Lambda(2l_t) - 2\kappa^\Lambda(l_t)}\right| \leq \frac{\sup_{x\in[\varepsilon,r]} |\kappa^\Lambda(xz)|}{\inf_{x\in[\varepsilon,r/2]} \left(\kappa^\Lambda(2x) - 2\kappa^\Lambda(x)\right)} \ .$$

Previuous supremum and infimum exist since $x \mapsto \kappa^\Lambda(zx)$ is continuous and it attains a maximum and a minimum on a compact interval. So, $D_{\varepsilon,r} \subset \mathcal{D}$ and Assumption 7 is verified because of Remark 2.5.11 4.

□

Remark 2.5.14. *1. Point 3. of Assumption 8 is also verified if we show that $2I_0 \subset D_{\varepsilon,r}$; in fact $D_{\varepsilon,r} \subset \mathcal{D}$ and*

$$\mathrm{supp}\Pi \cup (\mathrm{supp}\Pi + 1) \subset \frac{D_{\varepsilon,r}}{2} + \frac{D_{\varepsilon,r}}{2} \subset D_{\varepsilon,r},$$

because of Remark 2.5.11 4. and the fact that $D_{\varepsilon,r}$ is convexe.

2. From previous proof it follows that

$$\frac{d\kappa_t(z)}{d\rho_t} = \frac{\kappa^\Lambda(zl_t)}{\kappa^\Lambda(2l_t) - 2\kappa^\Lambda(l_t)} \ .$$

3. Admitting point 1. of Assumption 8, then $[0,T] \times I$ is compact. Since $t \mapsto \frac{d\kappa_t(z)}{d\rho_t}$ is continuous, point 4. of Assumption 8 would be verified.

We consider again the same class of options as in previous subsections. To conclude the verification of Assumption 8 it remains to show the following.

- I_0 is compact. This point will be trivially fulfilled.
- $2I_0 \subset D_{\varepsilon,r}$.

The only point to establish will be in fact

$$I \subset \{x|\varepsilon x, \frac{rx}{2} \in D_\Lambda\}. \tag{5.17}$$

Example 2.5.15. *1. $H = (S_T - K)_+$. Similarly to the case where X is a Lévy process, we take the second representation of the European Call. In this case $2I_0 = \{2R, 2\}$ and* (5.17) *is verified.*

2. $H = (K - S_T)_+$. Again, here $R < 0$, $2I_0 = \{2R\}$.
Again, we only have to require that D_Λ contains some negative values, which is the case for the two examples introduced in Remark 2.3.21. Selecting R in a proper way, (5.17) *is fulfilled.*

We provide now the FS decomposition and the solution to the minimization problem under Assumption 10. By Theorem 2.4.24 and Theorem 2.5.2, we obtain the following result.

Corollary 2.5.16. *We consider a process X of the form $X_t = \int_0^t l_s d\Lambda_s$ under Assumption 10. We consider an option H of the type* (4.28). *For $z \in \mathrm{supp}\Pi, t \in [0,T]$ we set*

$$\begin{aligned}
\lambda(s) &= \frac{\kappa^\Lambda(l_s)}{\kappa^\Lambda(2l_s) - 2\kappa^\Lambda(ls)}, \\
\gamma(z,s) &= \frac{\kappa^\Lambda((z+1)l_s) - \kappa^\Lambda(zl_s) - \kappa^\Lambda(l_s)}{\kappa^\Lambda(2l_s) - 2\kappa^\Lambda(l_s)}, \\
\eta(z,s) &= \kappa^\Lambda(zl_s) - \frac{\kappa^\Lambda(l_s)}{\kappa^\Lambda(2l_s) - 2\kappa^\Lambda(l_s)} \left(\kappa^\Lambda((z+1)l_s) - \kappa^\Lambda(zl_s) - \kappa^\Lambda(l_s)\right).
\end{aligned}$$

For convenience, if $z \notin \mathrm{supp}\Pi$ *then we define*

$$\gamma(z,\cdot) = \eta(z,\cdot) \equiv 0.$$

The following properties hold true.

1. *The FS decomposition is given by* $H_T = H_0 + \int_0^T \xi_t dS_t + L_T$ *where*

$$\begin{aligned} H_t &= \int_{\mathbb{C}} e^{\int_t^T \eta(z,ds)} S_t^z \Pi(dz), \\ \xi_t &= \int_{\mathbb{C}} \gamma(z,t) e^{\int_t^T \eta(z,ds)} S_{t-}^{z-1} \Pi(dz), \\ L_t &= H_t - H_0 - \int_0^t \xi_u dS_u. \end{aligned}$$

2. *The solution of the minimization problem is given by a pair* (V_0, φ) *where*

$$V_0 = H_0 \quad \textit{and} \quad \varphi_t = \xi_t + \frac{\lambda(t)}{S_{t-}}(H_{t-} - V_0 - G_{t-}(\varphi)).$$

2.5.4 A Log-Gaussian continuous process example.

Let (W_t) be a standard Brownian motion, we consider $X_t = W_{\psi(t)}$, where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly increasing function, including the pathological case where $\psi_t^{'} = 0$ a.e. For $z \in D = \mathbb{C}$, we have

$$\mathbb{E}[e^{zX_t}] = \mathbb{E}[e^{zW_{\psi(z)}}] = e^{\kappa_t(z)} = e^{\frac{z^2}{2}\psi(t)} \ ,$$

so that

$$\kappa_t(z) = \frac{z^2}{2}\psi(t) \ , \quad \rho(t) = \kappa_t(2) - 2\kappa_t(1) = \psi(t).$$

So

$$\langle M, M \rangle_t = \int_0^t S_{s-}^2 \psi(ds) \quad \text{and} \quad A_t = \int_0^t \frac{1}{2S_{s-}} d\langle M, M \rangle_s = \int_0^t \frac{1}{2} S_{s-}\psi(ds) \ ,$$

and the MVT process verifies

$$K_t = \int_0^t \frac{1}{4S_{s-}^2} d\langle M, M \rangle_s = \int_0^t \frac{1}{4}\psi(ds) = \frac{1}{4}\psi(t) \ .$$

Assumption 6 1. is verified since ψ is strictly increasing; Assumption 6 2., Assumption 7 and Assumption 8 are verified since $D = \mathcal{D} = \mathbb{C}$ and $\frac{d\kappa_t(z)}{d\rho_t} = \frac{z^2}{2}$ is continuous. Consequently all the conditions to apply Theorem 2.5.2 are satisfied and

$$\gamma(z,t) = z \ , \quad \eta(z,t) = \frac{\psi(t)}{2}(z^2 - z) \quad \text{and} \quad \lambda(t) \equiv \frac{1}{2}.$$

Hence we can compute the variance-optimal hedging strategy φ and the variance-optimal initial capital V_0 in this case

$$\varphi_t = \xi_t + \frac{1}{2S_{t-}}(H_{t-} - V_0 - \int_0^t \varphi_s dS_s)$$

and

$$H_t = \int_{\mathbb{C}} e^{\int_t^T \eta(z,ds)} S_t^z \Pi(dz) = \int_{\mathbb{C}} \exp\left\{\frac{z^2 - z}{2}(\Psi(T) - \Psi(t))\right\} S_t^z \Pi(dz)$$

$$\xi_t = \int_{\mathbb{C}} \gamma(z,t) e^{\int_t^T \eta(z,ds)} S_{t-}^{z-1} \Pi(dz) = \int_{\mathbb{C}} z \exp\left\{\frac{z^2 - z}{2}(\Psi(T) - \Psi(t))\right\} S_{t-}^{z-1} \Pi(dz)$$

Remark 2.5.17. *Calculating $\beta(y,z,t)$ of the quadratic error section, we find $\beta \equiv 0$. Therefore here also the quadratic error is zero. This confirms the fact that the market is **complete**, at least for the considered class of options.*

2.6 Application to Electricity

2.6.1 Hedging electricity derivatives with forward contracts

Electricity markets are composed by the Spot market setting prices for each delivery hour of the next day and the forward or futures market setting prices for more distant delivery periods. For simplicity, we will assume that interest rates are deterministic and zero so that futures prices are equivalent to forward prices. Forward prices given by the market correspond to a fixed price of one MWh of electricity for delivery in a given future period, typically a month, a quarter or a year. Hence, the corresponding term contracts are in fact swaps (i.e. forward contracts with delivery over a period) but are improperly named forward. However, the strong assumption that there are tradable forward contracts for all future time points $T_d \geq 0$ is usual and will be assumed here.

Because of non-storability of electricity, no dynamic hedging strategy can be performed on the spot market. Hedging instruments for electricity derivatives are then futures or forward contracts. The value of a forward contract offering the fixed price $F_0^{T_d}$ at time 0 for delivery of 1MWh at time T_d is by definition of the forward price, $S_0^{0,T_d} = 0$. Indeed, there is no cost to enter at time 0 the forward contract with the current market forward price $F_0^{T_d}$. Then, the value of the same forward contract S^{0,T_d} at time $t \in [0, T_d]$ is deduced by an argument of Absence of (static) Arbitrage as $S_t^{0,T_d} = e^{-r(T-t)}(F_t^{T_d} - F_0^{T_d})$. Hence, the dynamic of the hedging instrument $(S_t^{0,T_d})_{0 \leq t \leq T_d}$ is directly related (for deterministic interest rates) to the dynamic of forward prices $(F_t^{T_d})_{0 \leq t \leq T_d}$. Consequently, in the sequel we will focus on the dynamic of forward prices.

2.6.2 Electricity price models for pricing and hedging application

Observing market data, one can notice two main stylised features of electricity spot and forward prices:

- Volatility term structure of forward prices: the volatility increases when the time to maturity decreases;
- Non-Gaussianity of log-returns: log-returns can be considered as Gaussian for long-term contracts but they are clearly leptokurtic for short-term contratcs with huge spikes on the Spot market.

Hence, a challenge is to be able to describe with a single model, both the spikes on the short term and the volatility term structure of the forward curve. One reasonable attempt to do so is to consider the exponential Lévy factor model, proposed by Benth and Benth [11], or [21]. The forward price given at time t for delivery at time $T_d \geq t$, denoted $F_t^{T_d}$ is then modeled by a p-factors model, such that

$$F_t^{T_d} = F_0^{T_d} \exp(m_t^{T_d} + \sum_{k=1}^{p} X_t^{k,T_d}) \ , \quad \text{for all } t \in [0, T_d] \ , \text{where} \tag{6.18}$$

- $(m_t^{T_d})_{0 \leq t \leq T_d}$ is a real deterministic trend;
- For any $k = 1, \cdots p$, $(X_t^{k,T_d})_{0 \leq t \leq T_d}$ is such that $X_t^{k,T_d} = \int_0^t \sigma_k e^{-\lambda_k (T_d - s)} d\Lambda_s^k$, where $\Lambda = (\Lambda^1, \cdots, \Lambda^p)$ is a Lévy process on $\mathbb{R}^d$, with $\mathbb{E}[\Lambda_1^k] = 0$ and $Var[\Lambda_1^k] = 1$;
- $\sigma_k > 0$, $\lambda_k \geq 0$, are called respectively the *volatilities* and the *mean-reverting rates.*

Hence, forward prices are given as exponentials of PII with *non-stationary increments*. Then, the spot model is derived by setting $S_{T_d} = F^{T_d}_{T_d}$ and reduces to the exponential of a sum of possibly non-Gaussian Ornstein-Uhlenbeck processes. In practice, we consider the case of a one or a two factors model ($p = 1$ or 2), where the first factor X^1 is a non-Gaussian PII and the second factor X^2 is a Brownian motion with $\sigma_1 \gg \sigma_2$. Notice that this kind of model was originally developed and studied in details for interest rates in [64], as an extension of the Heath-Jarrow-Morton model where the Brownian motion has been replaced by a general Lévy process. Recent contributions in the subject are [33, 67].
Of course, this modeling procedure (6.18), implies incompleteness of the market. Hence, if we aim at pricing and hedging a European call on a forward with maturity $T \leq T_d$, it won't be possible, in general, to hedge perfectly the payoff $(F^{T_d}_T - K)_+$ with a hedging portfolio of forward contracts. Then, a natural approach could consist in looking for the variance optimal price and hedging portfolio. In this framework, the results of Section 2.4 generalizing the results of Hubalek & al in [49] to the case of non stationary PII can be useful. Similarly, some arithmetic models proposed in [8] for electricity prices, consists of replacing the right-hand side of (6.18) by its logarithm. Hence, with this kind of models the results of Section 2.3.4 can also be useful.

2.6.3 The non Gaussian two factors model

To simplify let us forget the upperscript T_d denoting the delivery period (since we will consider a fixed delivery period). We suppose that the forward price F follows the two factors model

$$F_t = F_0 \exp(m_t + X^1_t + X^2_t) \ , \quad \text{for all } t \in [0, T_d] \ , \text{where} \tag{6.19}$$

- m is a real deterministic trend starting at 0. It is supposed to be absolutely continuous with respect to Lebesgue;
- $X^1_t = \int_0^t \sigma_s e^{-\lambda(T_d - u)} d\Lambda_u$, where Λ is a Lévy process on $\mathbb{R}$ with Λ following a Normal Inverse Gaussian (NIG) distribution or a Variance Gamma (VG) distribution. Moreover, we will assume that $\mathbb{E}[\Lambda_1] = 0$ and $Var[\Lambda_1] = 1$;
- $X^2 = \sigma_l W$ where W is a standard Brownian motion on $\mathbb{R}$;
- Λ and W are independent.

- σ_s and σ_l standing respectively for the short-term volatilty and long-term volatility.

2.6.4 Verification of the assumptions

The result below helps to extend Theorem 2.5.2 to the case where X is a finite sum of independent PII semimartingales, each one verifying Assumptions 6, 7 and 8 for a given payoff $H = f(s_0 e^{X_T})$.

Lemma 2.6.1. *Let X^1, X^2 be two independent PII semimartingales with cumulant generating functions κ^i and related domains $D^i, \mathcal{D}^i, i = 1, 2$ characterized in Remark 2.3.8 and* (4.13)*. Let $f : \mathbb{C} \to \mathbb{C}$ of the form (4.28).*
For $X = X^1 + X^2$ with related domains $D, \mathcal{D}$ and cumulant generating function κ, we have the following.

1. *$D = D^1 \cap D^2$.*

2. *$\mathcal{D}^1 \cap \mathcal{D}^2 \subset \mathcal{D}$.*

3. *If X^1, X^2 verify Assumptions 6, 7 and 8, then X has the same property.*

Proof. Since X^1, X^2 are independent and taking into account Remark 2.3.8 we obtain 1. and

$$\kappa_t(z) = \kappa^1_t(z) + \kappa^2(z), \ \forall z \in D.$$

We denote by $\rho^i, i = 1, 2$, the reference variance measures defined in Remark 2.4.8. Clearly $\rho = \rho^1 + \rho^2$ and $d\rho^i \ll d\rho$ with $\|\frac{d\rho^i}{d\rho}\|_\infty \leq 1$.
If $z \in \mathcal{D}^1 \cap \mathcal{D}^2$, we can write

$$\begin{aligned}
\int_0^T \left|\frac{d\kappa_t(z)}{d\rho_t}\right|^2 d\rho_t &\leq 2\int_0^T \left|\frac{d\kappa^1_t(z)}{d\rho^1_t}\frac{d\rho^1_t}{d\rho_t}\right|^2 d\rho_t + 2\int_0^T \left|\frac{d\kappa^2_t(z)}{d\rho^2_t}\frac{d\rho^2_t}{d\rho_t}\right|^2 d\rho_t \\
&= 2\int_0^T \left|\frac{d\kappa^1_t(z)}{d\rho^1_t}\right|^2 \frac{d\rho^1_t}{d\rho_t} d\rho^1_t + 2\int_0^T \left|\frac{d\kappa^2_t(z)}{d\rho^2_t}\right|^2 \frac{d\rho^2_t}{d\rho_t} d\rho^2_t \\
&\leq 2\left(\int_0^T \left|\frac{d\kappa^1_t(z)}{d\rho^1_t}\right|^2 d\rho^1_t + \int_0^T \left|\frac{d\kappa^2_t(z)}{d\rho^2_t}\right|^2 d\rho^2_t\right).
\end{aligned}$$

This concludes the proof of $\mathcal{D}^1 \cap \mathcal{D}^2 \subset \mathcal{D}$ and therefore of the of Point 2.
Finally Point 3. follows then by inspection. □

With the two factors model, the forward price F is then given as the exponential of a PII, X, such that for all $t \in [0, T_d]$,

$$X_t = m_t + X_t^1 + X_t^2 = m_t + \sigma_s \int_0^t e^{-\lambda(T_d - u)} d\Lambda_u + \sigma_l W_t \ . \tag{6.20}$$

For this model, we formulate the following assumption.

Assumption 11. *1. $2\sigma_s \in D_\Lambda$.*

2. If $\sigma_l = 0$, we require Λ not to have deterministic increments.

3. We define $\varepsilon = \sigma_s e^{-\lambda T_d}, \quad r = 2\sigma_s$.

4. $f : \mathbb{C} \to \mathbb{C}$ is of the type (4.28) fulfilling (5.17)*.*

Proposition 2.6.2. *1. The cumulant generating function of X defined by (6.20), $\kappa : [0, T_d] \times D \to \mathbb{C}$ is such that for all $z \in D_{\varepsilon,r} := \{x \in \mathbb{R} \,|\, x\sigma_s \in D_\Lambda\} + i\mathbb{R}$, then for all $t \in [0, T_d]$,*

$$\kappa_t(z) = zm_t + \frac{z^2 \sigma_l^2 t}{2} + \int_0^t \kappa^\Lambda(z\sigma_s e^{-\lambda(T_d - u)}) du \ . \tag{6.21}$$

In particular for fixed $z \in D_{\varepsilon,r}$, $t \mapsto \kappa_t(z)$ is absolutely continuous with respect to Lebesgue measure.

2. Assumptions 6, 7 and 8 are verified.

Proof. We set $\tilde{X}^2 = m + X^2$. We observe that

$$D^2 = \mathcal{D}^2 = \mathbb{C}, \quad \kappa_t^2(z) = \exp(zm_t + z^2\sigma_l^2 \frac{t}{2}).$$

We recall that Λ and W are independent so that $\tilde{X}^2$ and X^1 are independent.
X^1 is a process of the type studied at Section 2.5.3. According to Proposition 2.5.13, Remark 2.5.14 and (5.17) it follows that Assumptions 6, 7 and 8 are verified for X^1.
Both statements 1. and 2. are now a consequence of Lemma 2.6.1. □

Remark 2.6.3. *For examples of f fulfilling* (5.17)*, we refer to Example 2.5.15.*

The solution to the mean-variance problem is provided by Theorem 2.5.2.

Theorem 2.6.4. *We suppose Assumption 11. The variance-optimal capital V_0 and the variance-optimal hedging strategy φ, solution of the minimization problem (2.1), are given by*

$$V_0 = H_0 \tag{6.22}$$

and the implicit expression

$$\varphi_t = \xi_t + \frac{\lambda_t}{S_{t-}}(H_{t-} - V_0 - \int_0^t \varphi_s dS_s), \tag{6.23}$$

where the processes (H_t),(ξ_t) and (λ_t) are defined as follows:

$$\begin{aligned}
\widetilde{z}_t : &= \sigma_s e^{-\lambda(T_d - t)}, \\
\gamma(z,t) : &= \frac{z\sigma_l^2 + \kappa^\Lambda((z+1)\widetilde{z}) - \kappa^\Lambda(z\widetilde{z}) - \kappa^\Lambda(\widetilde{z})}{\sigma_l^2 + \kappa^\Lambda(2\widetilde{z}) - 2\kappa^\Lambda(\widetilde{z})}, \\
\eta(z,t) : &= \left[zm_t + \frac{z^2\sigma_l^2}{2} + \kappa^\Lambda(z\widetilde{z}) - \gamma(z,t)\big(m_t + \frac{\sigma_l^2}{2} + \kappa^\Lambda(\widetilde{z})\big)\right] dt \ ,
\end{aligned}$$

$$\begin{aligned}
\lambda_t &= \frac{m_t + \frac{\sigma_l^2}{2} + \kappa^\Lambda(\widetilde{z})}{\sigma_l^2 + \kappa^\Lambda(2\widetilde{z}) - 2\kappa^\Lambda(\widetilde{z})}, \\
H_t &= \int_{\mathbb{C}} e^{\int_t^T \eta(z,ds)} S_t^z \Pi(dz), \\
\xi_t &= \int_{\mathbb{C}} \gamma(z,t) e^{\int_t^T \eta(z,ds)} S_{t-}^{z-1} \Pi(dz) \ .
\end{aligned}$$

The optimal initial capital is unique. The optimal hedging strategy $\varphi_t(\omega)$ is unique up to some $(P(d\omega) \otimes dt)$-null set.

Remark 2.6.5. *Previous formulae are practically exploitable numerically. The last condition to be checked is*

$$2\sigma_s \in D_\Lambda. \tag{6.24}$$

In our classical examples, this is always verified.

1. *Λ_1 is a Normal Inverse Gaussian random variable; if $\sigma_s \leq \frac{\alpha-\beta}{2}$ then (6.24) is verified.*
2. *Λ_1 is a Variance Gamma random variable then (6.24) is verified; if for instance $\sigma_s < \frac{-\beta+\sqrt{\beta^2+2\alpha}}{2}$.*

2.7 Simulations

2.7.1 Exponential Lévy

We consider the problem of pricing a European call, with payoff $(S_T - K)_+$, where the underlying process S is given as the exponential of a NIG Lévy process i.e. for all $t \in [0,T]$,

$$S_t = s_0 e^{X_t} \ , \quad \text{where } X \text{ is a Lévy process with } \ X_1 \sim NIG(\alpha, \beta, \delta, \mu) \ .$$

The time unit is the year and the interest rate is zero in all our simulations. The initial value of the underlying is $s_0 = 100$ Euros. The maturity of the option is $T = 0.25$ i.e. three months from now. Five different sets of parameters for the NIG distribution have been considered, going from the case of *almost Gaussian* returns corresponding to standard equities, to the case of *highly non Gaussian* returns. The standard set of parameters is estimated on the *Month-ahead base* forward prices of the French Power market in 2007:

$$\alpha = 38.46 \, , \ \beta = -3.85 \, , \ \delta = 6.40 \, , \ \mu = 0.64 \ . \tag{7.25}$$

Those parameters imply a zero mean, a standard deviation of 41%, a skewness (measuring the asymmetry) of -0.02 and an excess kurtosis (measuring the *fatness* of the tails) of 0.01. The other sets of parameters are obtained by multiplying parameter α by a coefficient C, (β, δ, μ) being such that the first three moments are unchanged. Note that when C grows to infinity the tails of the NIG distribution get closer to the tails of the Gaussian distribution. For instance, Table 2.1 shows how the excess kurtosis (which is zero for a Gaussian distribution) is modified with the five values of C chosen in our simulations.

Coefficient	$C = 0.08$	$C = 0.14$	$C = 0.2$	$C = 1$	$C = 2$
α	3.08	5.38	7.69	38.46	76.92
Excess kurtosis	1.87	0.61	0.30	0.01	$4.\,10^{-3}$

Figure 2.1: Excess kurtosis of X_1 for different values of α, (β, δ, μ) insuring the same three first moments.

We have compared on simulations the Variance Optimal strategy (VO) using the real NIG incomplete market model with the real values of parameters to the Black-Scholes strategy (BS) assuming Gaussian returns with the real values of mean and variance. Of course, the VO strategy is by definition theoritically optimal in continuous time, w.r.t. the quadratic norm. However, both strategies are implemented in discrete time, hence the performances observed in our simulations are spoiled w.r.t. the theoritical continuous rebalancing framework.

Strike impact on the pricing value and the hedging ratio

Figure 2.2 shows the initial capital (on the left graph) and the initial hedge ratio (on the right graph) produced by the VO and the BS strategies as functions of the strike, for three different sets of parameters $C = 0.08\,,\ C = 1\,,\ C = 2$. We consider $N = 12$ trading dates, which corresponds to operational practices on electricity markets, for an option expirying in three months. One can observe that BS results are very similar to VO results for $C \geq 1$ which corresponds to *almost Gaussian returns*. However, for small values of C, for $C = 0.08$, corresponding to highly non Gaussian returns, BS approach under-estimates *out-of-the-money* options and over-estimates *at-the-money* options. For instance, on Figure 2.3, one can observe that for $K = 99$ Euros the Black-Scholes Initial Capital (IC_{BS}) represents 122% of the Variance Optimal Initial Capital (IC_{VO}), while for $K = 150$ it represents only 57% of the variance optimal price. Moreover, the hedging strategy differs sensibly for $C = 0.08$, while it is quite similar to BS's ratio for $C \geq 1$.

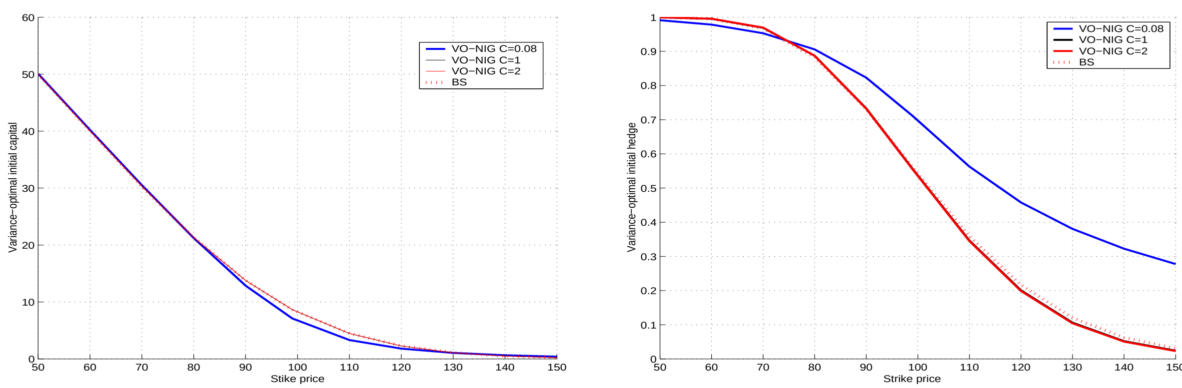


Figure 2.2: Initial capital (on the left) and hedge ratio (on the right) w.r.t. the strike, for $C = 0.08\,,\ C = 1\,,\ C = 2$.

Strikes	$K = 50$	$K = 99$	$K = 150$
IC_{VO}	50.08	7.11	0.40
IC_{BS} (vs IC_{VO})	50.00 (99.56%)	8.65 (121.73%)	0.23 (57.30%)

Figure 2.3: Initial Capital of VO pricing (IC_{VO}) vs Initial Capital of BS pricing (IC_{BS}) for $C = 0.08$.

Hedging error and number of trading dates

Figure 2.4 considers the hedging error (the difference between the terminal value of the hedging portfolio and the payoff) as a function of the number of trading dates, for a strike $K = 99$ Euros (at the money) and for five different sets of parameters C described on Figure 2.1. The bias (on the left graph) and standard deviation (on the right graph) of the hedging error have been estimated by Monte Carlo method on 5000 runs. Note that we could have used the formula stated in Theorem 2.5.4 to compute the variance of the error, but this would have give us the limiting error which does not take into account the additional error due to the finite number of trading dates.

In terms of standard deviation, the VO strategy seems to outperform sensibly the BS strategy, for small values of C. For instance, one can observe on Figure 2.5, for $C = 0.08$ that the VO strategy allows to reduce 10% of the standard deviation of the error. As expected, one can observe that the VO error converges to the BS error when C increases. This is due to the convergence of NIG log-returns to Gaussian log-returns when C increases (recall that the simulated log-returns are almost symmetric). One can distinguish two sources of incompleteness, the *rebalancing error* due to the dicrete rebalancing strategy and the *intrinsic error* due to the model incompleteness. On Figure 2.4, the hedging error (both for BS and VO) decreases with the number of trading dates and seems to converge to a limiting error corresponding to the intrinsic error. For $C = 1$ and for a small number of trading dates $N \leq 5$, the rebalancing error represents the most part of the hedging error, then it seems to vanish over $N = 30$ trading dates, where the intrinsic error is predominant. For small values of $C \leq 0.2$, even for small numbers of trading dates, the intrinsic error seems to be predominant. For $C \leq 0.2$ and $N \geq 12$ trading dates, it seems useless to increase the number of trading dates. Moreover, one can observe that for a small number of trading dates $N \leq 12$ and for large values of $C \geq 1$, BS seems to outperform the VO strategy, in terms of standard deviation. This can be interpreted as a consequence of the central limit theorem. Indeed, when the time between two trading dates increases the corresponding increments of the Lévy process converge to a Gaussian variable. Hence, the model error comitted by the BS approach decreases when the number of trading dates decreases.

In term of bias, the over-estimation of at-the-money options (observed for $C = 0.08$, on Figures 2.2, 2.3) seems to induce a positive bias for the BS error (see Figure 2.4), whereas the Bias of the VO error is negligeable (as expected from the theory).

However, to be more relevant in our analysis, we have compared on Figures 2.6 and 2.7, the performances of the BS hedging portfolio with the VO hedging portfolio starting with the same initial capital as the BS hedging portfolio. One can observe on Figure 2.6 that this approach allows to reduce the standard deviation of the VO hedging error (increasing the bias and of course the global quadratic error w.r.t. the VO strategy with optimal initial capital).

It is interesting to notice that, in terms of skewness and kurtosis, the VO strategy seems to outperform sensibly the BS strategy for small values of C. Figure 2.6 shows that for $C = 0.08$, the skewness of the BS hedging error is strongly negative (3 times greater than the VO error using the same initial capital) and the kurtosis is high (14 times greater than the VO error). Hence, in our simulations, BS strategy seems to imply more extreme losses than the VO strategy.

In conclusion, the VO approach provides initial capital and hedging strategies which are not significantly different from the BS approach except for log-returns with high excess kurtosis (with small values of parameter α in the NIG case). Similarly, we can observe (though the figures are not reported here) the same behaviour w.r.t. to the asymmetry of the distribution: the VO approach allows to outperform significantly the BS approach for strongly asymmetric log-returns (with high (absolute) values of parameter β in the NIG case). On the other hand, in more standard cases, the VO strategy seems to be comparable with the BS strategy in terms of quadratic error and to have the significant and unexpected advantage to limit extreme losses (skewness and kurtosis) compared to the BS strategy.

2.7.2 Exponential PII

We consider the problem of hedging and pricing a European call on an electricity forward, with a maturity $T = 0.25$ of three month. The maturity is equal to the delivery date of the forward contract $T = T_d$. As stated in Section 2.6, the natural hedging instrument is the corresponding forward contract with value $S_t^0 = e^{-r(T-t)}(F_t^T - F_0^T)$ for all $t \in [0, T]$, where

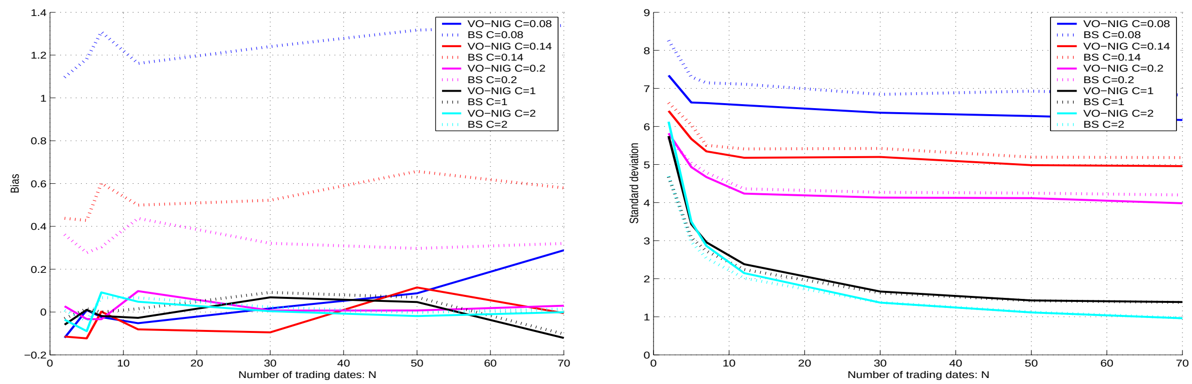


Figure 2.4: Hedging error w.r.t. the number of trading dates for different values of C and for $K = 99$ Euros (Bias, on the left and standard deviation, on the right).

Coefficient	$C = 0.08$	$C = 0.14$	$C = 0.2$	$C = 1$	$C = 2$
$\mathrm{Std}_{VO}/\mathrm{Std}_{BS}$	91.19%	95.88%	97.63%	107.52%	109.39%
$\mathrm{Bias}_{BS} - \mathrm{Bias}_{VO}$	1.20	0.57	0.32	0.022	0.019
$\mathrm{IC}_{BS} - \mathrm{IC}_{VO}$	1.55	0.7	0.39	0.01	0

Figure 2.5: Variance optimal hedging error vs Black-Scholes hedging error for different values of C and for $K = 99$ Euros (averaged values for different numbers of trading dates).

Moments	Mean	Standard deviation	Skewness	Kurtosis
VO	-0.049	6.59	-3.50	31.51
BS	1.27	7.25	-7.65	152.09
VO with $\mathrm{IC}_{VO} = \mathrm{IC}_{BS}$	1.39	6.47	-2.37	10.70

Figure 2.6: Empirical moments of the hedging error for $C = 0.08$ and $K = 99$ Euros (averaged values for different numbers of trading dates).

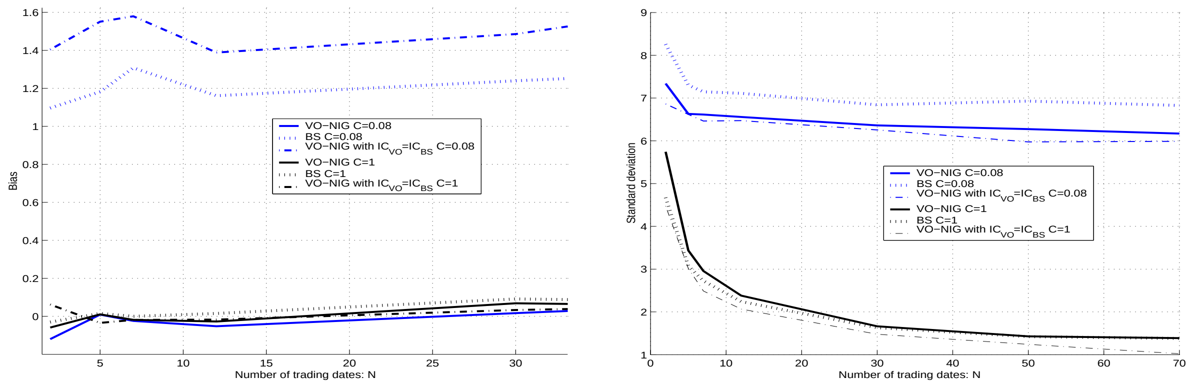


Figure 2.7: Hedging error of BS strategy v.s. the VO strategy with the same initial capital as BS w.r.t. the number of trading dates for different values of C and for $K = 99$ Euros (Bias, on the left and standard deviation, on the right).

$F^T = F$ is supposed to follow the NIG one factor model:

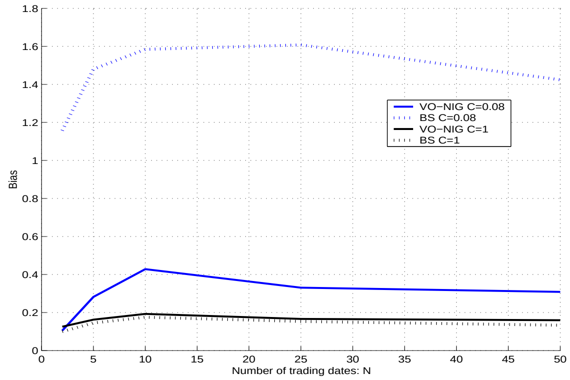
$$F_t = e^{X_t}\,, \quad \text{where } X_t = \int_0^t \sigma_s e^{-\lambda(T-u)} d\Lambda_u \quad \text{where } \Lambda \text{ is a Lévy process with} \quad \Lambda_1 \sim NIG(\alpha, \beta, \delta, \mu)\,.$$

The standard set of parameters ($C = 1$) for the distribution of Λ_1 is estimated on the same data as in the previous section (*Month-ahead base* forward prices of the French Power market in 2007):

$$\alpha = 15.81\,,\ \beta = -1.581\,,\ \delta = 15.57\,,\ \mu = 1.56\ .$$

Those parameters correspond to a standard and centered NIG distribution with a skewness of -0.019. The estimated annual short-term volatility and mean-reverting rate are $\sigma_s = 57.47\%$ and $\lambda = 3$. The other sets of parameters considered in simulations are obtained by multiplying parameter α by a coefficient C, (β, δ, μ being such that the first three moments are unchanged).

Figure 2.8 shows the Bias and Standard deviation of the hedging error as a function of the number of trading dates estimated by Monte Calo method on 5000 runs. The results are comparable to those obtained in the case of the Lévy process, on Figure 2.8. However, one can notice that the BS strategy does no more outperform the VO strategy for small numbers of trading dates as observed in the Lévy case. This is due to the fact that X_t is no more a sum of i.i.d. variables.



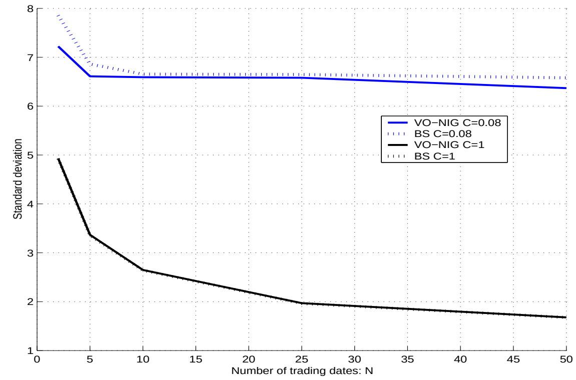


Figure 2.8: Hedging error w.r.t. the number of trading dates for $C = 0.08$ and $C = 1$, for $K = 99$ Euros (Bias, on the left and standard deviation, on the right).

Moments	Mean	Standard deviation	Skewness	Kurtosis
VO	0.43	6.59	−2.89	16.24
BS	1.58	6.65	−3.79	25.53

Figure 2.9: Empirical moments of the hedging error for $C = 0.08$, $N = 10$ and $K = 99$ Euros.

Chapter 3

Variance-Optimal hedging in discrete time

This chapter is the object of the paper [46].

Abstract. *We consider the discretized version of a (continuous-time) two-factor model introduced by Benth and coauthors for the electricity markets. For this model, the underlying is the exponent of a sum of independent random variables. We provide and test an algorithm, which is based on the celebrated Föllmer-Schweizer decomposition for solving the mean-variance hedging problem. In particular, we establish that decomposition explicitely, for a large class of vanilla contingent claims. Interest is devoted in the choice of rebalancing dates and its impact on the hedging error, regarding the payoff regularity and the non stationarity of the log-price process.*

Key words: Variance-optimal hedging, Föllmer-Schweizer decomposition, Lévy process, Cumulative generating function, Characteristic function, Normal Inverse Gaussian distribution, Electricity markets, Incomplete Markets, Processes with independent increments, trading dates optimization.

2010 AMS-classification: 60G50, 60G51, 91G10, 60J05, 62M99

JEL-classification: C02, C15, G11, G12, G13

3.1 Introduction

It is well known that the classical Black-Scholes model does not allow in real applications to replicate perfectly contingent claims. Of course, this is due to market incompleteness and specifically two major reasons : the non-Gaussianity of prices log-returns and the finite number of trading dates. The impact of these features have been intensively studied separately in the literature.

There is a large literature on pricing and hedging with non Gaussian models (allowing for stochastic volatility or jumps), in a continuous time setup. Then, the hedging error related to the dicretization of the hedging strategy is in general ignored or investigated separately. One popular approach is the Variance-Optimal hedging: if H denotes the payoff of the option and S^c denotes the underlying price process, the goal is to minimize the mean squared hedging error

$$\mathbb{E}[(V_T - H)^2] \quad \text{with} \quad V_T = c + \int_0^T v_t dS_t^c \ .$$

over all initial endowments $c \in \mathbb{R}$ and all (in some sense) admissible strategies v. The first paper specifically on this subject is due to Duffie and Richardson, see [30]. Among significant early contributions there are [72, 73, 76, 66, 41], a fairly complete recent article on the structure of mean-variance hedging, with a rich bibliography is provided by [17]. One of the now classical tools is the so called Föllmer-Schweizer decomposition. Given a square integrable r.v. H and an $(\mathcal{F}_t)$-semimartingale $S = (S_t)_{t\geq 0}$, that decomposition consists in finding a triple (H_0, ξ, L) where H_0 is $\mathcal{F}_0$-measurable, ξ is $(\mathcal{F}_t)$-predictable and L is a martingale being orthogonal to the martingale part M of S such that $H = H_0 + \int_0^T \xi_s dS_s + L_T$. In the recent years, some attention was focused on finding explicit or quasi explicit formulae for the Föllmer-Schweizer decomposition or the optimal strategy for the mean-variance hedging problem. For instance [9] gave an expression based on Clark-Ocone type decompositions related to Lévy type measures when the underlying is a Lévy martingale, [23] still in the martingale case with techniques of partial integro differential equations. [49] obtained significant explicit decompositions when the underlying is the exponential of a Lévy process and the contingent claim is a vanilla type option appearing as some generalized Laplace transform of a finite complexe measure. Other significant semi-explicite formulae appear in [54, 55]. [49] was continued in chapter 2 of this thesis in the framework of processes with independent increments with some applications to the electricity market.

However, in practice, the hedging strategy cannot be implemented continuously and the

resulting optimal strategy has to be discretized. Hence, to be really relevant the hedging error should take into account this further approximation.
An alternative approach, less investigated in the literature, is to consider directly the hedging problem in discrete time as proposed by Cox Ross and Rubinstein [24]. The first incomplete market analysis in the spirit of minimizing a quadratic risk is due to [35]. They worked with the so-called local risk-minimisation. The problem of Variance-Optimal hedging in the discrete time setup was proposed in [70, 74]. In the recent years some interest on discrete time was rediscovered in [13, 15, 56]. [18] revisits the seminal paper [35] in the spirit of global risk minimization. In the discrete-time context, a significant role was played by the analogous of the previously mentioned FS-decomposition. It is recalled in Definition 3.2.8.
Recently, many approaches have been proposed to obtain explicit or quasi-explicit formulas for computing both the variance optimal trading strategies and hedging errors in discrete time. For instance, in [4], Angelini and Herzel derive closed formulas for the variance optimal hedge ratio and the corresponding hedging error variance when the underlying asset is a geometric Brownian motion which is martingale. As we said, Kallsen and co-authors contributed at providing semi-explicit formulae for the Variance-Optimal hedging problem both in discrete and continuous time, for various kind of models. In particular in [49], semi-explicit formula are derived for the (discrete and continuous time) Variance-Optimal hedging strategy and for the resulting hedging error, in the specific case where the logarithm of the underlying price is a process with stationary independent increments. One major idea proposed in [49]and [16] consists in expressing the payoff as a linear combination of exponential payoffs for which the variance optimal hedging strategy can be expressed explicitly. With a similar methodology and in the same setting, Angelini and Herzel [5] determine the Laplace transform of the variance of the error produced by a standard delta hedging strategy when applied to several class of models. In [28] similar results are provided in the continuous time setup. In this paper, we use the generalized Laplace transform approach to extend the results of [49] to the case of processes with independent increments (PII) relaxing the stationary assumption on log-returns. The semi-explicite discrete Föllmer-Schweizer decomposition is stated in Proposition 3.3.11, the solution to the mean-variance hedging problem in Theorem 3.4.1. The expression of the quadratic hedging error in Theorem 3.4.3 gives a priori a criterion of market completeness as far as vanilla options are concerned. This confirms that the (even not stationary) binomial model is complete, see Proposition 3.4.5.

Our discrete time model consists in fact in the discretization of continuous time models which are exponential of processes of independent increments. Given a continuous-time model $(S^c_t)_{t\geq 0}$ (the superscript c referring to the $c$ontinuous time setting), where $S^c_t = s_0 \exp(X^c_t)$ and X^c is a process with independent increments and discrete trading dates $t_0, t_1, \cdots, t_N$, our discrete model will be $S = (S_k)$, such that $S_k = S^c_{t_k}$, for all $k = 0, 1, \cdots N$. In this dicrete time setting, the Variance-Optimal pricing and hedging problem consists in looking for the initial endowments $c \in \mathbb{R}$ and the admissible strategy $v = (v_k)$ which minimizes

$$\mathbb{E}[(V_T^N - H)^2] \quad \text{with} \quad V_T^N = c + \sum_{k=1}^{N} v_k \Delta S_k \ .$$

This framework is indeed well suited to take into account together both the non-Gaussianity of log-returns and hedging errors due to the discreteness of trading times. Our investigation for quasi-explicit formulae when the underlying is the exponential of sums of independent random variables is due to two reasons.

1. The first one comes from the fact that the basic continuous time model can be time-inhomogeneous in a natural way, see for instance chapter 2 of this thesis.

2. The second, more original reason, is that the discretized times, which correspond in our case to the rebalancing dates, are not necessarily uniformly chosen.

First, some prices exhibit non stationary and non-Gaussian log-returns. One common example of this phenomena can be observed on electricity futures or forward market: the forwards volatility increases when the time to delivery decreases whereas the tails of log-returns distribution get heavier resulting in huge spikes on the Spot. The exponential Lévy factor model, proposed in [11] and [21] allows to represent both the volatility term structure and the spikes on the short term. More precisely, the forward price given at time t for delivery of 1MWh at time $T_d \geq t$, denoted $F^{T_d}_t$ is then modeled by a two factors model, such that

$$S^c_t := F^{T_d}_t = F^{T_d}_0 \exp(m^{T_d}_t + \int_0^t \sigma_S e^{-\lambda(T_d - u)} d\Lambda_u + \sigma_L W_t) \ , \quad \text{for all } t \in [0, T_d] \ , \qquad (1.1)$$

where m is a real deterministic trend, Λ a real Lévy process and W a real Brownian motion. Hence, forward prices are modeled as exponentials of PII with *non-stationary increments* and existing results from [49] valid for stationary independent processes cannot be applied for that kind of models.

Another announced motivation for our developement is to be able to analyse the impact of a non-homogeneous discretization of the trading dates on the Variance-Optimal hedging error.

The issue of considering non-homogeneous trading dates was first considered by Geiss S. in [39] and Geiss S, Geiss C. in [40] who analysed the impact on the hedging error of discretizing a continuously rebalanced hedging portfolio. He showed that for a given irregular payoff (e.g. a digital call), concentrating rebalancing dates near the maturity instead of rebalancing regularly can improve the convergence rate of the hedging error. Still in the continuous time setup, recently, Gobet and Makhlouf [42] provided precise results quantifying the impact of the choice of rebalancing dates on the convergence rate of the hedging error regarding the payoff regularity. Hence, it seems to be of real interest to be able to consider such non-homogeneous grids. However, if the continuous time log-price model $X^c = \log(S^c) - \log(s_0)$ has independent and stationary increments, considering non-homogeneous trading dates involves a non stationary discrete time process X such that $X_k = X^c_{t_k}$ for $k = 0, \cdots N$, where $t_0, t_1, \cdots, t_N$ denote the non-homogeneous trading dates. Hence, here again existing results from [49] cannot be applied neither for hedging at non-homogeneous times nor for evaluating the resulting hedging error.

In the present work, we have performed some numerical tests concerning both applications. One major observation is the remarkable robustness of the Black-Scholes strategy that still achieves quasi-minimal hedging errors variances, with both non Gaussian log-returns and discrete rebalancing dates. Besides, our tests show that when hedging with electricity forward contracts, the impact of the choice of the rebalancing dates on the hedging error seems to be more important than the choice of log-returns distribution (Gaussian or Normal Inverse Gaussian, in our case). Concerning the case of hedging an irregular payoff (a digital call, in our case), our numerical tests confirm the result of [39]. In *almost Gaussian cases*, we observe that the variance optimal hedging error, can be noticeably reduced by optimizing the rebalancing dates. However, this phenomena is less pronounced when the tails of the log-returns distribution get heavier for which the hedging error gets less sensitive to the rebalancing grid. This suggests that the result of [39] and [42] could not be extended straightforwardly to the non Gaussian case.

This article is organized as follows. In Section 2, notations and generalities on the discrete Föllmer-Schweizer decomposition are presented. In Section 3, we derive semi-explicit Föllmer-Schweizer decomposition for exponential of PII. Section 4 is devoted to the solution to the global minimization problem. Illustrative example and simulation results are given in Section 5; in particular, subsection 3.5.2 is concerned with data comimg from the electricity market.

3.2 Generalities and Discrete Föllmer-Schweizer decomposition

We present the context of the problem studied by [74]. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $N \in \mathbb{N}^*$ a fixed natural number and $\mathbb{F} = (\mathcal{F}_k)_{k=1,\cdots,N}$ a filtration. We shall assume that $\mathcal{F} = \mathcal{F}_N$. Let (S_k) be a real-valued, $\mathbb{F}$-adapted, square-integrable process. We denote by ΔS_k the increments $S_k - S_{k-1}$, for $k = 1, \cdots, N$. We use the convention that a sum (respectively product) over an empty set is zero (resp. one).

Definition 3.2.1. *We denote by Θ the set of all predictable processes v (i.e.: v_k is $\mathcal{F}_{k-1}$-measurable for each $k \geq 1$) such that $v_k \Delta S_k \in \mathcal{L}^2(\Omega)$ for $k = 1, \cdots, N$. For $v \in \Theta$, $G(v)$ is the process defined by*

$$G_k(v) := \sum_{j=1}^{k} v_j \Delta S_j, \quad \text{for} \quad k = 1, \cdots, N.$$

The problem addressed in [74] is the following.
Given $H \in \mathcal{L}^2(\Omega)$, we look for (V_0^*, φ^*) which minimize the quantity

$$\mathbb{E}\left[(H - V_0 - G_T(\varphi))^2\right] \ , \tag{2.2}$$

over $V_0 \in \mathbb{R}$ and $\varphi \in \Theta$. It will be called **discrete time optimization problem**. The expression $\mathbb{E}\left[(H - V_0^* - G_T(\varphi^*))^2\right]$ will be called the **variance optimal hedging error**.

Definition 3.2.2. *Schweizer [74] introduces the following **non-degeneracy condition (ND)**. We say that S satisfies the non-degeneracy condition (ND) if there exists a constant $\delta \in]0, 1[$ such that*

$$(\mathbb{E}[\Delta S_k | \mathcal{F}_{k-1}])^2 \leq \delta \mathbb{E}[(\Delta S_k)^2 | \mathcal{F}_{k-1}] \ ,$$

P.a.s for $k = 1, \cdots, N$.

Remark 3.2.3.

1. *If (S_k) is an $\mathbb{F}$-martingale then (ND) is always verified.*

2. *Note that by Jensen's inequality, we always have*

$$(\mathbb{E}[\Delta S_k | \mathcal{F}_{k-1}])^2 \leq \mathbb{E}[(\Delta S_k)^2 | \mathcal{F}_{k-1}] \quad \text{a.s.}$$

The point of condition (ND) is to ensure a strict inequality uniformly in ω.

To obtain another formulation of (ND), we now express S in its Doob decomposition as $S_k = M_k + A_k$ where M_k is a square-integrable $(\mathcal{F}_k)$-martingale and A_k is a square-integrable predictable process with $A_0 = 0$. It is well-known that this decomposition is unique and is given through

$$\Delta A_k := \mathbb{E}[\Delta S_k | \mathcal{F}_{k-1}] \ , \quad \text{and} \quad \Delta M_k := \Delta S_k - \Delta A_k \ .$$

We will operate with the help of some conditional moments and conditional variance setting

$$Var[\Delta S_k | \mathcal{F}_{k-1}] := \mathbb{E}[(\Delta S_k)^2 | \mathcal{F}_{k-1}] - \mathbb{E}[\Delta S_k | \mathcal{F}_{k-1}]^2 \ .$$

Remark 3.2.4. *For $k = 1, \ldots, N$, we have the following.*

1. $\mathbb{E}[(\Delta S_k)^2 | \mathcal{F}_{k-1}] = \mathbb{E}[(\Delta M_k)^2 | \mathcal{F}_{k-1}] + (\Delta A_k)^2$;
2. $Var[\Delta S_k | \mathcal{F}_{k-1}] = \mathbb{E}[(\Delta M_k)^2 | \mathcal{F}_{k-1}]$;
3. *Previous conditional variance vanishes if and only if* $\Delta M_k = 0$.

We introduce the predictable process λ_k by

$$\lambda_k := \frac{\Delta A_k}{\mathbb{E}[(\Delta S_k)^2 | \mathcal{F}_{k-1}]} = \frac{\mathbb{E}[\Delta S_k | \mathcal{F}_{k-1}]}{\mathbb{E}[(\Delta S_k)^2 | \mathcal{F}_{k-1}]} \ , \tag{2.3}$$

for all $k = 1, \cdots, N$. These quantities could be theoretically infinite.

Remark 3.2.5. *Suppose that $P(\Delta S_k = 0) = 0$ for any $k = 1, \cdots, N$.*

1. *Then $\mathbb{E}[(\Delta S_k)^2 | \mathcal{F}_{k-1}] > 0$ a.s. In fact, let $B = \{\omega | \mathbb{E}[(\Delta S_k)^2(\omega) | \mathcal{F}_{k-1}] = 0\}$. This implies $\Delta A_k = 0$ on B because of Remark 3.2.4 1. By the same Remark,*

   $$0 = 1_B \mathbb{E}[(\Delta M_k)^2 | \mathcal{F}_{k-1}] = \mathbb{E}[1_B (\Delta M_k)^2 | \mathcal{F}_{k-1}] \ ,$$

   so $\Delta M_k = 0$ on B. This implies that $\Delta S_k = 0$ a.s. on B. By assumption, B is forced to be a null set.

2. *Previous point 1. guarantees in particular that (λ_k) are all finite.*

Definition 3.2.6. *The* ***mean-variance tradeoff process*** *of S is defined by*

$$K_j^d := \sum_{l=1}^{j} \frac{\mathbb{E}[\Delta S_l | \mathcal{F}_{l-1}]^2}{Var[\Delta S_l | \mathcal{F}_{l-1}]} \ ,$$

for all $j = 1, \cdots, N$. K^d is the discrete version of the continuous time corresponding process K defined for instance in Definition 2.2.11 of chapter 2 of this thesis or in Section 1. of [72].

Proposition 3.2.7. *The condition (ND) is fulfilled if and only if*

$$\frac{\mathbb{E}[\Delta S_k|\mathcal{F}_{k-1}]^2}{Var[\Delta S_k|\mathcal{F}_{k-1}]}$$

is a.s. bounded uniformly in ω and k.

Proof. See (1.6) in [74]. □

A basic tool for solving the optimization problem (2.2) in [74] is the discrete Föllmer-Schweizer decomposition.

Definition 3.2.8. *Denote by $S = M + A$ the Doob decomposition of S into a martingale M and a predictable process A. A complex-valued square integrable random variable H is said to admit a* ***discrete Föllmer-Schweizer decomposition*** *(or simply discrete FS-decomposition) if there exists a $\mathcal{F}_0$-measurable H_0, a complex-valued process ξ such that both $Re\xi(z), Im\xi(z)$ belong to Θ, and a square integrable $\mathbb{C}$-valued martingale L^H such that*

1. *$L^H M$ is an $\mathbb{F}$-martingale;*
2. *$E(L_0^H) = 0$,*
3. *$H = H_0 + \sum_{k=1}^N \xi_k \Delta S_k + L_N^H$.*

When Point 1. is fulfilled L^H and M are called **strongly orthogonal***.*
If H is a real valued r.v. then H admits a **real discrete FS decomposition** *if it admits a FS decomposition with $H_0 \in \mathbb{R}$ and ξ being a real valued process. In this case $\xi \in \Theta$.*

3.2.1 Existence and structure of an optimal strategy

Assumption 12. *$(S_k)_{k=1,\cdots,N}$ satisfies the nondegeneracy condition (ND).*

Remark 3.2.9.
1. *Under Assumption 12, Proposition 2.6 of [74] guarantees that every square integrable real random variable H admits a real discrete FS-decomposition.*
2. *That decomposition is unique because of Remark 4.11 of [70].*
3. *Previous two points imply the existence and uniqueness of the discrete Föllmer-Schweizer decomposition when H is complex square integrable random variable.*

4. *An immediate consequence is that the decomposition of a real square integrable random variable is necessarily real.*

Other tools for solving the optimization problem and evaluating the error are the following.

Proposition 3.2.10. *If S satisfies (ND), then $G_N(\Theta)$ is closed in $\mathcal{L}^2(P)$.*

Proof. See [74], Theorem 2.1. □

Theorem 3.2.11. *Suppose that $S = M + A$ has a deterministic mean-variance tradeoff process. Let H be a square integrable real random variable with discrete real FS- decomposition given by $H = H_0 + G_N(\xi^H) + L_N^H$.*

1. *The optimization problem (2.2) is solved by (V_0^*, φ^*) where $V_0^* = H_0$ and φ^* is determined by*

$$\varphi_k^* = \xi_k^H + \lambda_k(H_{k-1} - H_0 - G_{k-1}(\varphi^*)).$$

2. *Suppose that $\mathcal{F}_0$ is a trivial σ-field. The hedging error is given by*

$$J_0 = \sum_{k=1}^{N} \mathbb{E}[(\Delta L_k^H)^2] \prod_{j=k+1}^{N} (1 - \lambda_j \Delta A_j).$$

Proof. Point 1. follows from Proposition 4.3 of [74]. Concerning Point 2., $L_0^H = 0$ a.s. since $\mathcal{F}_0$ is trivial. The result follows from Theorem 4.4 of [74]; □

Similarly to [49], we will calculate it explicitely in the case where S is the exponential of process with independent increments.

3.3 Exponential of PII processes

From now on, we will suppose that $(X_n)_{n=1,\cdots,N}$ is a sequence of random variables with **independent increments**, i.e. $(X_1 - X_0, \cdots, X_N - X_{N-1})$ are independent random variables. From now on, without restriction of generality, it will not be restrictive to suppose $X_0 = 0$. We also define the process $(S_n)_{n=1,\cdots,N}$ as $S_n = s_0 \exp(X_n)$, $0 \le n \le N$ for some $s_0 > 0$.

Definition 3.3.1. *We denote $D = \{z \in \mathbb{C} | \exp(zX_N) \in \mathcal{L}^1\}$.*

3.3.1 Discrete cumulant generating function

Definition 3.3.2. *We define the **discrete cumulant generating function** as $m : D \times \{0, \cdots, N\} \to \mathbb{C}$ with $m(z,n) = \mathbb{E}[e^{z\Delta X_n}]$ for all $n = 1, \cdots, N$ and by convention $m(z,0) \equiv 1$.*

This function is a discrete version of the cumulant generating function investigated in the previos chapter, chapter 2, of this thesis.

Remark 3.3.3. *1. If $z \in D$ then the property of independent increments implies that $m(z,n) = \mathbb{E}[\exp(z\Delta X_n)]$ is well-defined for all $z \in D$ and $n = 0, 1, \cdots, N$.*

2. If $\gamma \in \mathbb{R}^+ \cap D$, Cauchy-Schwarz inequality implies that $[0,\gamma] + i\mathbb{R} \subset D$; if $\gamma \in \mathbb{R}^- \cap D$ then $[\gamma, 0] + i\mathbb{R} \subset D$. This shows in particular that D is convexe.

Remark 3.3.4. *When X has stationary increments then we have $m(z,n) = m(z,1)$ for all $n = 1, \cdots, N$. We denote this quantity by $m(z)$ similarly as in [49], Section 2.*

We formulate some assumptions which are analogous to those in continuous time case, see chapter 2 of this thesis.

Assumption 13. *1. ΔX_n is never deterministic for every $n = 1, \cdots, N$.*

2. $2 \in D$.

Remark 3.3.5. *In particular, $S_n \in \mathcal{L}^2(\Omega)$, for every $n = 0, 1, \cdots, N$, because $2 \in D$.*

Lemma 3.3.6. *$z \mapsto m(z,n)$ is continuous for any $n = 0, 1, \cdots, N$. In particular, if K is a compact real set then $\sup_{z \in K + i\mathbb{R}} |m(z,n)| < \infty$.*

Proof. We set $Y = \Delta X_n$ for fixed $n \in \{1, \cdots, N\}$. Let $z \in D$ and (z_p) be a sequence converging to z. Obviously $\exp(z_p Y) \to \exp(zY)$ a.s. In order to conclude we need to show that the sequence $(\exp(z_p Y))$ is uniformly integrable. After extraction of subsequences, we can separately suppose that

1. either $\min_n Re(z_p) \leq Re(z_p) \leq Re(z)$,

2. or $\max_n Re(z_p) \geq Re(z_p) \geq Re(z)$.

This implies the existence of $a, A \in D \cap \mathbb{R}$ such that $a \leq Re(z_p) \leq A$, for all $p \in \mathbb{N}$. Consequently if $M > 0$, for every $p \in \mathbb{N}$, we have

$$\mathbb{E}[\exp(z_p Y) 1_{|Y|>M}] \leq \int_{-\infty}^{-M} \exp(yRe(z_p)) d\mu_Y(y) + \int_M^{\infty} \exp(yRe(z_p)) d\mu_Y(y)$$

where μ_Y is the distribution law of Y. Previous sum is bounded by

$$\int_{-\infty}^{-M} \exp(ay) d\mu_Y(y) + \int_M^{\infty} \exp(Ay) d\mu_Y(y)$$

Since M is arbitrarly big, the result is established. □

Lemma 3.3.7. *Let $n = 0, \cdots, N$.*

1. $\mathbb{E}[e^{\Delta X_n} - 1]^2 = m(2, n) - 2m(1, n) + 1$.
2. $Var[e^{\Delta X_n} - 1] = m(2, n) - m(1, n)^2$.
3. $\mathbb{E}[e^{\Delta X_n} - 1] = m(1, n) - 1$.

Proof. Statements 1. and 3. follow in elementary manner using the definition of m. Statement 2. follows from statement 1. and the fact that $\mathbb{E}[e^{\Delta X_n} - 1] = m(1, n) - 1$. □

Remark 3.3.8. *$m(2, n) - m(1, n)^2$ is strictly positive for any $n = 1, \cdots, N$. In fact Assumption 13 1. implies that $e^{\Delta X_n} - 1$ is never deterministic.*

Remark 3.3.9. *For $z \in D$ and $n \in \{1, \cdots N\}$, we have*

$$\mathbb{E}(S_n^z) = s_0^z \prod_{k=1}^n m(z, k).$$

Proposition 3.3.10. *For $n \in \{1, \cdots N\}$, we have*

1. $\Delta A_n = \mathbb{E}[\Delta S_n | \mathcal{F}_{n-1}] = (m(1, n) - 1) S_{n-1}$.
2. $Var[\Delta S_n | \mathcal{F}_{n-1}] = (m(2, n) - m(1, n)^2) S_{n-1}^2$.
3. *Condition (ND) is always satisfied.*
4. $$\lambda_n = \frac{1}{S_{n-1}} \frac{m(1, n) - 1}{m(2, n) - 2m(1, n) + 1}.$$

5. *The mean-variance tradeoff process K^d is deterministic.*

Proof. 1. follows from $\mathbb{E}[\Delta S_n|\mathcal{F}_{n-1}] = S_{n-1}\mathbb{E}[e^{\Delta X_n} - 1]$ and Lemma 3.3.7 3.

2. Since
$$\mathbb{E}[(\Delta S_n)^2|\mathcal{F}_{n-1}] = S_{n-1}^2\mathbb{E}[(e^{\Delta X_n} - 1)^2], \tag{3.4}$$
we can write
$$\begin{aligned} Var[\Delta S_n|\mathcal{F}_{n-1}] &:= \mathbb{E}[(\Delta S_n)^2|\mathcal{F}_{n-1}] - \mathbb{E}[\Delta S_n|\mathcal{F}_{n-1}]^2 \ , \\ &= S_{n-1}^2\mathbb{E}[(e^{\Delta X_n} - 1)^2] - S_{n-1}^2\mathbb{E}[e^{\Delta X_n} - 1]^2 \\ &= S_{n-1}^2 Var[e^{\Delta X_n} - 1]. \end{aligned}$$
The conclusion follows from Lemma 3.3.7 2.

3. We make use of Proposition 3.2.7. In our context we have
$$\frac{\mathbb{E}[\Delta S_n|\mathcal{F}_{n-1}]^2}{Var[\Delta S_n|\mathcal{F}_{n-1}]} = \frac{(m(1,n) - 1)^2}{m(2,n) - m(1,n)^2}. \tag{3.5}$$
The denominator of the right-hand side never vanishes because of Remark 3.3.8.

4. It follows from (2.3), (3.4), Lemma 3.3.7 1. and point 1. of this Proposition.

5. It is a consequence of point 3. and Definition 3.2.6.

□

3.3.2 Discrete Föllmer-Schweizer decomposition

Similarly to [49] and the previous chapter of this thesis, chapter 2, we would like to obtain the discrete Föllmer-Schweizer decomposition of a random variable of the type $H = S_N^z$, for some sutable $z \in \mathbb{C}$. The proposition below generalizes Lemma 2.4 of [49].

Proposition 3.3.11. *Under Assumption 13, let $z \in D$ fixed, such that $2Re(z) \in D$. Then $H(z) = S_N^z$ admits a discrete Föllmer-Schweizer decomposition*
$$\begin{cases} H(z)_n &= H(z)_0 + \sum_{k=1}^n \xi(z)_k \Delta S_k + L(z)_n \\ H(z)_N &= H(z) = S_N^z \end{cases}$$

where

$$\begin{aligned}
H(z)_n &= h(z,n)S_n^z \ , \quad \textit{for all} \quad n \in \{0, \cdots N\} \\
\xi(z)_n &= g(z,n)h(z,n)S_{n-1}^{z-1} \ , \quad \textit{for all} \quad n \in \{1, \cdots N\} \\
L(z)_n &= H(z)_n - H(z)_0 - \sum_{k=1}^{n} \xi(z)_k \Delta S_k \ , \quad \textit{for all} \quad n \in \{0, \cdots N\}
\end{aligned} \tag{3.6}$$

and $g(z,n)$*,* $h(z,n)$ *are defined by*

$$h(z,n) := \prod_{i=n+1}^{N} (m(z,i) - g(z,i)[m(1,i) - 1]) \tag{3.7}$$

$$g(z,n) := \frac{m(z+1,n) - m(1,n)m(z,n)}{m(2,n) - m(1,n)^2} \tag{3.8}$$

Remark 3.3.12. *1.* $z+1 \in D$ *because* D *is convexe, taking into account Assumption 13 2.*

2. If $2Rez$ *does not belong to* D*, for simplicity, we will set*

$$g(z,n) \equiv h(z,n) \equiv H(z)_n \equiv \xi(z)_n \equiv L(z)_n \equiv 0.$$

3. If K *is a compact real interval, for any* $n \in \{0, \cdots N\}$ *we have* $\sup_{z \in K + i\mathbb{R}}(|g(z,n)| + |h(z,n)|) < \infty$*.*

Remark 3.3.13. *Suppose that* $(X_n)_{n=0,\cdots,N}$ *is a* **process with stationary increments** *i.e. such that* $(X_1 - X_0, \cdots, X_N - X_{N-1})$ *are identically distributed random variables. According to Remark 3.3.4, we have*

$$g(z,n) = \frac{m(z+1) - m(1)m(z)}{m(2) - m(1)^2}. \tag{3.9}$$

We will denote in this case $g(z)$ *the right-hand side of (3.9). Moreover* $h(z,n) = h(z)^{N-n}$ *where*

$$h(z) = m(z) - g(z)[m(1) - 1]. \tag{3.10}$$

Proof of Proposition 3.3.11. Since $z+1 \in D$ all the involved expressions are-well defined. Since $L(z)_0 = 0$, we need to prove the following.

1. $L(z)$ is an $\mathbb{F}$-square integrable martingale.

2. $(L(z)M)$ is an $\mathbb{F}$-martingale.

From (3.6), it follows that

$$\Delta L(z)_n = L(z)_n - L(z)_{n-1} = h(z,n)S_n^z - h(z,n-1)S_{n-1}^z - g(z,n)h(z,n)S_{n-1}^z(e^{\Delta X_n} - 1);$$

$L(z)_n$ is square integrable for any $n \in \{0, \cdots, N\}$ since $2z \in D$ and (X_n) has independent increments.
Since $S_n^z = S_{n-1}^z e^{z\Delta X_n}$, we have

$$\Delta L(z)_n = S_{n-1}^z \left[h(z,n)e^{z\Delta X_n} - h(z,n-1) - g(z,n)h(z,n)(e^{\Delta X_n} - 1)\right] \,, \tag{3.11}$$

therefore

$$\mathbb{E}[\Delta L(z)_n|\mathcal{F}_{n-1}] = S_{n-1}^z \mathbb{E}\left[h(z,n)e^{z\Delta X_n} - h(z,n-1) - g(z,n)h(z,n)(e^{\Delta X_n} - 1)\right].$$

1. To show that $L(z)$ is a martingale it is enough to show that

$$\mathbb{E}\left[h(z,n)e^{z\Delta X_n} - h(z,n-1) - g(z,n)h(z,n)(e^{\Delta X_n} - 1)\right] = 0.$$

   Previous expression implies the relation $h(z,n)m(z,n) - h(z,n-1) - g(z,n)h(z,n)(m(1,n) - 1) = 0$ for any $0 \leq n \leq N$ which is equivalent to

$$h(z,n-1) = h(z,n)\left(m(z,n) - g(z,n)(m(1,n) - 1)\right)$$

   for any $0 \leq n \leq N$.
   Previous backward relation with $h(z,N) = 1$ leads to (3.7).

2. It remains to prove that $(L(z)_n M_n)$ is a martingale. Since $L(z)_n$ and M_n are square integrable for any n then $L(z)_n M_n \in \mathcal{L}^1$. We prove now that $\mathbb{E}[\Delta L(z)_n \Delta M_n|\mathcal{F}_{n-1}] = 0$. Proposition 3.3.10 1. implies that the Doob decomposition $S = M + A$ of S satisfies

$$\Delta A_n = (m(1,n) - 1)S_{n-1}\ .$$

   Moreover

$$\Delta M_n = \Delta S_n - \Delta A_n = S_{n-1}(e^{\Delta X_n} - 1) - S_{n-1}(m(1,n) - 1) = S_{n-1}(e^{\Delta X_n} - m(1,n)).$$

   Coming back to (3.11)

$$\Delta L(z)_n \Delta M_n = S_{n-1}^{z+1}(e^{\Delta X_n} - m(1,n))\left[h(z,n)e^{z\Delta X_n} - h(z,n-1) - g(z,n)h(z,n)(e^{\Delta X_n} - 1)\right].$$

Taking the conditional expectation with respect to $\mathcal{F}_{n-1}$, we obtain

$$\begin{aligned}
\mathbb{E}[\Delta L(z)_n \Delta M_n|\mathcal{F}_{n-1}] &= \mathbb{E}[S_{n-1}^{z+1}(e^{\Delta X_n} - m(1,n)) \\
&\quad \left[h(z,n)e^{z\Delta X_n} - h(z,n-1) - g(z,n)h(z,n)(e^{\Delta X_n}-1)\right]|\mathcal{F}_{n-1}] \\
&= S_{n-1}^{z+1}\mathbb{E}[(e^{\Delta X_n} - m(1,n)) \\
&\quad \left[h(z,n)e^{z\Delta X_n} - h(z,n-1) - g(z,n)h(z,n)(e^{\Delta X_n}-1)\right]] \\
&= S_{n-1}^{z+1}\mathbb{E}[e^{(z+1)\Delta X_n}h(z,n) \\
&- e^{\Delta X_n}h(z,n-1) - e^{\Delta X_n}g(z,n)h(z,n)(e^{\Delta X_n}-1) \\
&- m(1,n)h(z,n)e^{z\Delta X_n} + m(1,n)h(z,n-1) \\
&+ m(1,n)g(z,n)h(z,n)(e^{\Delta X_n}-1)].
\end{aligned}$$

Again by Lemma 3.3.7, previous quantity equals zero if and only if

$$h(z,n)m(z+1,n) - g(z,n)h(z,n)m(2,n) - m(1,n)h(z,n)m(z,n) + m(1,n)^2 g(z,n)h(z,n) = 0 ,$$

or equivalently

$$m(z+1,n) - g(z,n)m(2,n) - m(1,n)m(z,n) + m(1,n)^2 g(z,n) = 0.$$

Remark 3.3.8 finally shows that the right-hand side must have the form (3.8). This concludes the proof of Proposition 3.3.11.

□

3.3.3 Discrete Föllmer-Schweizer decomposition of special contingent claims

We consider now options $f : \mathbb{C} \to \mathbb{R}$ as in the first chapter of this thesis in continuous time, 2, of the type

$$H = f(S_N) , \quad \text{with} \quad f(s) = \int_{\mathbb{C}} s^z \Pi(dz) , \tag{3.12}$$

where Π is a (finite) complex measure in the sense of Rudin [68], Section 6.1. An integral representation of some basic European calls can be found in chapter 2 of this thesis or[49]. The European Call option $H = (S_T - K)_+$ and Put option $H = (K - S_T)_+$ have a representation of the form (4.28) provided by the lemma below.

Lemma 3.3.14. *Let $K > 0$.*

1. *For arbitrary $0 < R < 1$, $s > 0$, we have*

$$(s-K)_+ - s = \frac{1}{2\pi i}\int_{R-i\infty}^{R+i\infty} s^z \frac{K^{1-z}}{z(z-1)} dz \ . \tag{3.13}$$

2. *For arbitrary $R < 0$, $s > 0$*

$$(K-s)_+ = \frac{1}{2\pi i}\int_{R-i\infty}^{R+i\infty} s^z \frac{K^{1-z}}{z(z-1)} dz \ . \tag{3.14}$$

We need at this point an assumption which depends on the support of Π. We set $I_0 := \mathrm{supp}\Pi \cap \mathbb{R}$.

Assumption 14.

1. *I_0 is compact.*
2. *$2I_0 \subset D$.*

Remark 3.3.15.

1. *Assumption 14 is always verified (for any $0 < R < 1$) for the Call since $I_0 = \{R, 1\}$ is always included in $[0, 1]$ which is a subset of $\frac{D}{2}$ by Assumption 13 1.*
2. *Assumption 14 is also verified for the Put, choosing suitable R provided that D contains some negative values.*

Remark 3.3.16.

1. *Since D is convexe, Assumption 14 2. and the fact that $2 \in D$ imply that $I_0 + 1 \subset D$.*
2. *Since I_0 is compact, taking $\Pi = \delta_z$ for some $z \in \mathbb{C}$, Assumption 14 is equivalent to the assumptions of Proposition 3.3.11.*
3. *Since I_0 is compact, Assumption 13 point 1. and Lemma 3.3.6 imply that $\sup_{z \in 2I_0 + i\mathbb{R}} |m(z,n)| < \infty$, for every $n = 1, \cdots, N$.*
4. *Taking into account Remark 3.3.12 and points 2. and 3. we also get $\sup_{z\in\mathbb{C}}(|g(z,n)| + |h(z,n)|) < \infty$, for every $n = 1, \cdots, N$.*

Remark 3.3.17. *Notice that Assumption 14 is relatively weak and verified for a large class of models, whereas Assumption 8 required in the first chapter of this thesis, 2, to derive similar results, in the continuous time setting, noticeably restricts the set of underlying dynamics.*

Lemma 3.3.18. *For any $n \in \{0, \cdots, N\}$, according to the notations of Proposition 3.3.11 we have*

1. $\sup_{z\in\mathbb{C}} \mathbb{E}[|H(z)_n|^2] < \infty$;
2. $\sup_{z\in\mathbb{C}} \mathbb{E}[|\xi(z)_n|^2(\Delta S_n)^2] < \infty$, *for* $n \geq 1$;
3. $\sup_{z\in\mathbb{C}} \mathbb{E}[(\Delta L(z)_n)^2] < \infty$.

Proof. Remark 3.3.5, together with point 4. of Remark 3.3.16 show the validity of point 1. Point 3. is a consequence of points 1 and 2. Concerning this last point, let $n \in \{1, \cdots, N\}$. By Lemma 3.3.7 1.

$$\begin{aligned}
\mathbb{E}[|\xi^H(z)_n|^2(\Delta S_n)^2] &= g(z,n)^2 h(z,n)^2 \mathbb{E}(S_{n-1}^{2z})(m(2,n) - 2m(1,n) + 1) \\
&= g(z,n)^2 h(z,n)^2 m(2z, n-1)(m(2,n) - 2m(1,n) + 1)
\end{aligned}$$

The conclusion follows by Remark 3.3.16. □

Proposition below extends Proposition 2.5 of [49].

Proposition 3.3.19. *We suppose the validity of Assumptions 13 and 14. Any contingent claim $H = f(S_N)$ admits the real discrete FS decomposition H given by*

$$\begin{cases} H_n &= H_0 + \sum_{k=1}^n \xi_k^H \Delta S_k + L_n^H \\ H_N &= H \end{cases}$$

where

$$H_n = \int_{\mathbb{C}} H(z)_n \Pi(dz) \tag{3.15}$$

$$\xi_n^H = \int_{\mathbb{C}} \xi(z)_n \Pi(dz) \tag{3.16}$$

$$L_n^H = \int_{\mathbb{C}} L(z)_n \Pi(dz) = H_n - H_0 - \sum_{k=1}^n \xi_k^H \Delta S_k, \tag{3.17}$$

according to the same notations as in Proposition 3.3.11 and Remark 3.3.12. Moreover the processes (H_n),(ξ_n^H) and (L_n^H) are real-valued.

Proof. We proceed similarly to [49], Proposition 2.1. We need to prove that L^H (resp. $L^H M$) is a square integrable (resp. integrable) martingale. This will follow from Propostion 3.3.11 and Fubini's theorem. The use of Fubini's is justified by Lemma 3.3.18. The fact that H, ξ^H and L are real processes follows from Remark 3.2.9 4. □

3.4 The solution of the minimization problem

3.4.1 Mean-Variance Hedging

We can now summarize the solution to the optimization problem.

Theorem 3.4.1. *We suppose the validity of Assumptions 13 and 14. Let $H = f(S_N)$ with discrete real FS-decomposition*

$$\begin{cases} H_n &= H_0 + \sum_{k=1}^{n} \xi_k^H \Delta S_k + L_n^H \\ H_N &= H. \end{cases}$$

A solution to the optimal problem (2.2) is given by (V_0^*, φ^*) with $V_0^* = H_0$ and φ^* is determined by

$$\varphi_n^* = \xi_n^H + \lambda_n \left(H_{n-1} - H_0 - \sum_{i=1}^{n-1} \varphi_i^* \Delta S_i \right) \tag{4.18}$$

where λ_n is defined for all $n \in \{1, \cdots, N\}$, by

$$\lambda_n = \frac{1}{S_{n-1}} \frac{m(1,n) - 1}{m(2,n) - 2m(1,n) + 1} \ . \tag{4.19}$$

Moreover the solution is unique (up to a null set).

Remark 3.4.2. *In the case that X has stationary increments, we obtain*

$$\lambda_n = \frac{1}{S_{n-1}} \frac{m(1) - 1}{m(2) - 2m(1) + 1} \ ,$$

where $m(n) = E(\exp(nX_1))$. This confirms the results of Section 2. in [49].

Proof of theorem 3.4.1. The existence follows from Theorem 3.2.11, Proposition 3.3.19 and Proposition 3.3.10 points 3., 4. and 5.
Uniqueness follows exactly as in the proof of Proposition 2.5 of [49]: in our case Lemma 3.3.7 gives

$$Var[e^{\Delta X_n} - 1] = m(2,n) - m(1,n)^2.$$

□

3.4.2 The Hedging Error

The hedging error is given by Theorem 3.2.11 since the mean-tradeoff process is deterministic.

Theorem 3.4.3. *We suppose the validity of Assumptions 13 and 14. The variance of the hedging error in Theorem 3.4.1 equals*

$$J_0 = \int_{\mathbb{C}} \int_{\mathbb{C}} J_0(y,z) \Pi(dy) \Pi(dz) \ , \tag{4.20}$$

with

$$J_0(y,z) = \begin{cases} s_0^{y+z} \sum_{k=1}^N b(y,z;k) h(z,k) h(y,k) \prod_{\ell=2}^k m(y+z,\ell-1) \prod_{j=k+1}^N a(j) & : \quad y,z \in \mathrm{supp}\pi \\ 0 & : \quad \text{otherwise} \end{cases} \tag{4.21}$$

where

$$a(j) = \frac{m(2,j) - m(1,j)^2}{m(2,j) - 2m(1,j) + 1}$$

and

$$b(y,z;k) = \frac{\rho(y,z;k)\rho(1,1;k) - \rho(y,1;k)\rho(z,1;k)}{\rho(1,1;k)} \ , \tag{4.22}$$

where $\rho(y,z;k) = m(y+z;k) - m(y,k)m(z,k)$, $y,z \in \mathrm{supp}\Pi$.

Remark 3.4.4. *The function* ρ *above plays an analogous role to the complex valued function with the same name introduced in chapter 2 at Definition 2.4.3 in the continuous time framework.*

Proof. We proceed again similarly to the proof of theorem 2.1 of [49]. Theorem 3.2.11 gives that the hedging error is given by

$$J_0 = \sum_{k=1}^N \mathbb{E}[(\Delta L_k^H)^2] \prod_{j=k+1}^N (1 - \lambda_j \Delta A_j) \ . \tag{4.23}$$

Proposition 3.3.10 gives

$$\begin{aligned} \Delta A_j &= \mathbb{E}[\Delta S_j | \mathcal{F}_{j-1}] = (m(1,j) - 1) S_{j-1} \\ \lambda_j &= \frac{1}{S_{j-1}} \frac{m(1,j) - 1}{m(2,j) - 2m(1,j) + 1}, \end{aligned} \tag{4.24}$$

so

$$1 - \lambda_j \Delta A_j = a(j), \tag{4.25}$$

and it remains to calculate $\mathbb{E}[(\Delta L_k^H)^2]$. Since

$$\Delta L_k^H = \int_{\mathbb{C}} \Delta L(z)_k \Pi(dz)$$

we have

$$(\Delta L_k^H)^2 = \int_{\mathbb{C}} \int_{\mathbb{C}} \Delta L(y)_k \Delta L(z)_k \Pi(dy)\Pi(dz) \tag{4.26}$$

and hence by Fubini's Theorem

$$\mathbb{E}[(\Delta L_k^H)^2] = \int_{\mathbb{C}} \int_{\mathbb{C}} \mathbb{E}[\Delta L(y)_k \Delta L(z)_k] \Pi(dy)\Pi(dz).$$

Relation (3.11) says that

$$\begin{aligned}\Delta L(z)_k &= S_{k-1}^{y+z} \left[h(y,k) e^{y\Delta X_k} - h(y,k-1) - g(y,k)h(y,k)(e^{\Delta X_k} - 1) \right] \\ &\quad \left[h(z,k) e^{z\Delta X_k} - h(z,k-1) - g(z,k)h(z,k)(e^{\Delta X_k} - 1) \right].\end{aligned}$$

Taking the expectation we obtain

$$\begin{aligned}\mathbb{E}[\Delta L_k(y)\Delta L_k(z)] &= \mathbb{E}[S_{k-1}^{y+z}]\{(h(z,k)h(y,k)m(y+z,k) - h(z,k)h(y,k-1)m(z,k) \\ &- h(z,k)h(y,k)g(y,k)\mathbb{E}[e^{z\Delta X_k}(e^{\Delta X_k} - 1)] - h(z,k-1)h(y,k)m(y,k) \\ &+ h(z,k-1)h(y,k-1) + h(z,k-1)h(y,k)g(y,k)\mathbb{E}[e^{\Delta X_k} - 1] \\ &- h(z,k)h(y,k)g(z,k)\mathbb{E}[e^{y\Delta X_k}(e^{\Delta X_k} - 1)] + h(z,k)h(y,k-1)g(z,k)\mathbb{E}[e^{\Delta X_k} - 1] \\ &+ h(z,k)h(y,k)g(z,k)g(y,k)\mathbb{E}[(e^{\Delta X_k} - 1)^2]\}\end{aligned}$$

Recalling that $\mathbb{E}[(e^{\Delta X_k} - 1)^2] = m(2,k) - 2m(1,k) + 1$ and $\mathbb{E}[e^{\Delta X_k} - 1] = m(1,k) - 1$, we obtain

$$\begin{aligned}\mathbb{E}[\Delta L_k(y)\Delta L_k(z)] &= \mathbb{E}[S_{k-1}^{y+z}]\{(h(z,k)h(y,k)m(y+z,k) - h(z,k)h(y,k-1)m(z,nk \\ &- h(z,k)h(y,k)g(y,k)(m(z+1,k) - m(z,k)) - h(z,k-1)h(y,k)m(y,k) \\ &+ h(z,k-1)h(y,k-1) + h(z,k-1)h(y,k)g(y,k)(m(1,k) - 1) \\ &- h(z,k)h(y,k)g(z,n)(m(y+1,k) - m(y,k)) \\ &+ h(z,k)h(y,k-1)g(z,k)(m(1,n) - 1) \\ &+ h(z,k)h(y,k)g(z,k)g(y,k)(m(2,k) - 2m(1,k) + 1)\}.\end{aligned} \tag{4.27}$$

By Proposition 3.3.11 we have

$$\begin{aligned} h(y,k-1) &= h(y,k)[m(y,k)-g(y,k)(m(1,k)-1)] \\ h(z,k-1) &= h(z,k)[m(z,k)-g(z,k)(m(1,k)-1)]. \end{aligned} \tag{4.28}$$

We replace the right-hand sides of (4.28) in (4.27) and we factorize by $h(z,k)h(y,k)$. Finally, after simplification we obtain

$$\begin{aligned} \mathbb{E}[\Delta L_k(y)\Delta L_k(z)] &= \mathbb{E}[S_{k-1}^{y+z}]h(z,k)h(y,k)\{m(y+z,k) \\ &- m(z,k)m(y,k)+m(z,k)g(y,k)m(1,k)+m(y,k)g(z,k)m(1,k) \\ &- g(y,k)m(z+1,k)-g(z,k)m(y+1,k) \\ &- g(z,k)g(y,k)[m(1,k)-1]^2 \\ &+ g(z,k)g(y,k)[m(2,k)-2m(1,k)+1]\}. \end{aligned}$$

Hence,

$$\mathbb{E}[\Delta L_k(y)\Delta L_k(z)] = \mathbb{E}[S_{k-1}^{y+z}]h(z,k)h(y,k)\tilde{b}(y,z;k), \tag{4.29}$$

where

$$\mathbb{E}[S_{k-1}^{y+z}] = s_0^{y+z}\mathbb{E}[e^{(y+z)\Delta X_{k-1}}] = s_0^{y+z}\prod_{\ell=2}^{k} m(y+z,\ell-1) \tag{4.30}$$

and

$$\begin{aligned} \tilde{b}(y,z,k) &= \{m(y+z,k)-m(z,k)m(y,k)-g(y,k)m(z+1,k)-g(z,k)m(y+1,k) \\ &+ m(z,k)g(y,k)m(1,k)+m(y,k)g(z,k)m(1,k) \\ &- g(z,k)g(y,k)m(1,k)^2+g(z,k)g(y,k)m(2,k)\}. \end{aligned}$$

We observe that

$$\tilde{b}(y,z,k) = \rho(y,z;k)-g(y,k)\rho(z,1;k)-g(z,k)\rho(y,1,k)+g(y,k)g(z,k)\rho(1,1,k). \tag{4.31}$$

Since,

$$g(y,k) = \frac{\rho(y,1;k)}{\rho(1,1;k)}$$

$$g(z,k) = \frac{\rho(z,1;k)}{\rho(1,1;k)}$$

it follows that $\tilde{b}(y,z,k) = b(y,z,k)$.
Finally, (4.24), (4.25), (4.26), (4.29), (4.30) and (4.31) give

$$\begin{aligned} J_0(y,z) &= s_0^{y+z} \sum_{k=1}^{N} b(y,z,k)h(z,k)h(y,k) \prod_{\ell=2}^{k} m(y+z,\ell-1) \prod_{j=k+1}^{N} (1-\lambda_j \Delta A_j) \\ &= s_0^{y+z} \sum_{k=1}^{N} b(y,z,k)h(z,k)h(y,k) \prod_{\ell=2}^{k} m(y+z,\ell-1) \prod_{j=k+1}^{N} a(j). \end{aligned}$$

□

From the expression of the hedging error variance (4.21), we can derive a sort of criterion for completeness for market asset pricing models. More precisely, the condition

$$b(y,z;k) = 0 \ , \quad \text{for all } y,z \in D \text{ and } k \in \{1, \cdots N\} \tag{4.32}$$

characterizes the prices models that are exponential of PII for which every payoff (that can be written as an inverse Laplace transform) can be hedged. In the specific case of a Binomial (even inhomogeneous) model, we retrieve the fact that $J_0(y,z) \equiv 0$ and so $J_0 = 0$. In fact, that model is complete.

Proposition 3.4.5. *Let $a, b \in \mathbb{R}$, $X_k = a$ with probability p_k and $X_k = b$ with probability $(1-p_k)$. Then $J_0(y,z) \equiv 0$ for every $y,z \in \frac{D}{2}$.*

Proof. Writing $p = p_k$, $k \in \{0,1,\cdots,N\}$, we have

$$\begin{aligned} \rho(y,z;k) &= pe^{a(y+z)} + (1-p)e^{b(y+z)} - (pe^{ay} + (1-p)e^{by})(pe^{az} + (1-p)e^{bz}) \\ &= p(1-p)\left(e^{a(y+z)} + e^{b(y+z)} + e^{by+az} + e^{bz+ay}\right) \\ &= p(1-p)\left(e^{ay} + e^{by}\right)\left(e^{az} + e^{bz}\right). \end{aligned}$$

So

$$\rho(y,z;k)\rho(1,1;k) = p^2(1-p)^2\left(e^{ay} + e^{by}\right)\left(e^{az} + e^{bz}\right)\left(e^{a} + e^{b}\right)^2$$

On the other hand, this obviously equals $\rho(y,1;k)\rho(z,1;k)$. □

If X is a process with stationary and independent increments we reobtain the result of [49]].

Proposition 3.4.6. *Let(X_k) be a process with stationary increments. We denote*

$$\begin{aligned} m(y) &:= \mathbb{E}(\exp(yX_1)) \\ \mathcal{H}(y) &:= m(y) - \frac{m(1)-1}{m(2)-m(1)^2}(m(y+1)-m(1)m(y)) \\ a &:= \frac{m(2)-m(1)^2}{m(2)-2m(1)+1}. \end{aligned}$$

Then

$$J_0 = \int_{\mathbb{C}}\int_{\mathbb{C}} J_0(y,z)\Pi(dy)\Pi(dz)$$

with

$$J_0(y,z) = \begin{cases} s_0^{y+z}\beta(y,z)\frac{a(y,z)^N - m(y+z)^N}{a(y,z)-m(y+z)}, & if \quad a(y,z) \neq m(y+z) \\ s_0^{y+z}\beta(y,z)Nm(y+z)^{N-1} & if \quad \mathrm{a(y,z) = m(y+z)} \end{cases}. \tag{4.33}$$

where

$$\begin{aligned} a(y,z) &= a\mathcal{H}(y)\mathcal{H}(z), \\ \beta(y,z) &= m(y+z) - \frac{m(2)m(y)m(z) - m(1)m(y+1)m(z) - m(1)m(y)m(z+1) + m(y+1)m(z+1)}{m(2)-m(1)^2}. \end{aligned}$$

Proof. We observe that for $k \in \{0, \cdots, N\}$, we have

$$\begin{aligned} m(y+z,k) &= m(y+z) \\ h(y,k) &= \mathcal{H}(y)^{N-k} \\ h(z,k) &= \mathcal{H}(z)^{N-k}. \end{aligned}$$

So

$$\prod_{j=k+1}^{N} a(j) = \left(\frac{m(2)-m(1)^2}{m(2)-2m(1)+1}\right)^{N-k} = a^{N-k}$$

Consequently, expression (4.21) for $y, z \in supp(\Pi)$,

$$J_0(y,z) = s_0^{y+z}\beta(y,z)\sum_{k=1}^{N} m(y+z)^{k-1}\left(\mathcal{H}(y)\mathcal{H}(z)a\right)^{N-k}$$

$$J_0(y,z) = \begin{cases} s_0^{y+z}\beta(y,z)\frac{(m(y+z)-\mathcal{H}(y)\mathcal{H}(z)a)^N}{m(y+z)-a\mathcal{H}(y)\mathcal{H}(z)}, & : if \quad m(y+z) \neq a\mathcal{H}(y)\mathcal{H}(z) \\ s_0^{y+z}\beta(y,z)Nm(y+z)^{N-1} & : if \quad m(y+z) = a\mathcal{H}(y)\mathcal{H}(z) \end{cases}. \tag{4.34}$$

This concludes the proof of the proposition. ∎

3.5 Numerical results

As announced in the introduction, we will now apply the quasi-expicit formulae derived in previous sections to measure the impact of the choice of the rebalancing dates on the hedging error. We will consider two cases that motivated the present work:

1. the underlying continuous time log-price model has stationary increments but the payoff to hedge is irregular, such as a **Digital call**, so that, as shown in [39, 42], hedging near the maturity can improve the hedge;
2. the payoff is regular (e.g. classical call) but the underlying continuous time model shows a volatility term structure which is exponentially increasing near the maturity, such as **electricity forward prices**. For this reason it seems again judicious to hedge more frequently near the maturity, where the volatility accelerates.

3.5.1 The case of a Digital option

We consider the problem of hedging and pricing a Digital call, with payoff $f(s) = \mathbf{1}_{[K,\infty)}(s)$ of maturity $T > 0$. From (35) in [49], the payoff of this option can be expressed as

$$f(s) = \lim_{c\to\infty} \frac{1}{2\pi i} \int_{R-ic}^{R+ic} s^z \frac{K^{-z}}{z} dz \ , \tag{5.35}$$

for an arbitrary $R > 0$. This implies that the complex measure Π is given by

$$\Pi(dz) = \frac{1}{2\pi i} \frac{K^{-z}}{z} dz \ . \tag{5.36}$$

However, such measure is only σ-finite so that application of Theorem 3.4.1 is not rigourously valid. Nevertheless, using improper integrals one should be able to recover the main statement. In this section, this will be assumed so that formula (4.20) will be used in the case of a Digital option.

The underlying process S^c is given as the exponential of a Normal Inverse Gaussian Lévy process (see Appendix 3.5.2) i.e. for all $t \in [0, T]$,

$$S^c_t = e^{X^c_t} \ , \quad \text{where } X^c \text{ is a Lévy process with } \ X^c_1 \sim NIG(\alpha, \beta, \delta, \mu) \ .$$

Given $N+1$ discrete dates $0 = t_0 < t_1 < \cdots < t_N = T$, we associate the discrete model pricing $X = X^N$ where $X_k = X^c_{t_k}, k \in \{0, \ldots, N\}$. X is a discrete time process with indepedent increments. The related cumulant generating function $z \mapsto m(z,k)$ associated to the increment $\Delta X_k = X_k - X_{k-1} = X^c_{t_k} - X^c_{t_{k-1}}$ for $k \in \{1, \cdots N\}$ is defined on $D = [-\alpha - \beta; \alpha - \beta]$. We refer for this to chapter 2 Remark 2.3.21 2., since X^c is a NIG process. By additivity we can show that

$$m(z,k) = \mathbb{E}[\exp(z\Delta X_k)] = \exp\Big(\Delta t_k\big[\mu z + \delta\big(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta+z)^2}\big)\big]\Big) \qquad (5.37)$$

for $z \in D, k \in \{0, \ldots, N\}$.
For other informations on the NIG law, the reader can refer to Appendix 3.5.2.
Assumption 13 1. is trivially verified, Assumption 13 2. is verified as soon as

$$2 \leq \alpha - \beta$$

Thanks to Remark 3.3.15 Assumption 14 is automatically verified for the call and put representations given by Lemma 2.4.26, and, by similar arguments, even for the digital option. The time unit is the year and the interest rate is zero in all our tests. The initial value of the underlying is $s_0 = 100$ Euros. The maturity of the option is $T = 0.25$ i.e. three months from now. Four different sets of parameters for the NIG distribution have been considered, going from the case of *almost Gaussian* returns corresponding to standard equities, to the case of *highly non Gaussian* returns. The standard set of parameters is estimated on the *Month-ahead base* forward prices of the French Power market in 2007:

$$\alpha = 38.46\,,\ \beta = -3.85\,,\ \delta = 6.40\,,\ \mu = 0.64\ . \qquad (5.38)$$

Those parameters imply a zero mean, a standard deviation of 41%, a skewness (measuring the asymmetry) of -0.02 and an excess kurtosis (measuring the *fatness* of the tails) of 0.01. The other sets of parameters are obtained by multiplying parameter α by a coefficient C, (β, δ, μ) being such that the first three moments are unchanged. Note that when C grows to infinity the tails of the NIG distribution get closer to the tails of the Gaussian distribution. For instance, Table 2.1 shows how the excess kurtosis (which is zero for a Gaussian distribution) is modified with the four values of C chosen in our tests. We compute the Variance Optimal (VO) hedging error given by (4.20), for different grids of rebalancing dates. The corresponding initial capital V_0 denoted by $V_0^* = H_0$ in Theorem 3.4.1 is computed using Proposition 3.3.19.

In particular, we consider the parametric grid introduced in [39], [40] and [42]

$$\pi^{b,N} := \{0 = t_0^{b,N}, t_1^{b,N}, \cdots, t_N^{b,N}\}$$

Coefficient	$C = 0.14$	$C = 0.2$	$C = 1$	$C = 2$
α	5.38	7.69	38.46	76.92
Excess kurtosis	0.61	0.30	0.01	$4.\,10^{-3}$

Figure 3.1: Excess kurtosis of X_1 for different values of α, (β, δ, μ) insuring the same three first moments.

defining, for any real $b \in (0, 1]$, N rebalancing dates such that

$$t_k^{b,N} = T - T(1 - \frac{k}{N})^{1/b} \quad \text{for all } k \in \{0, \cdots, N-1\} \; . \tag{5.39}$$

Note that $\pi^{1,N}$ coincides with equidistant rebalancing dates whereas when b converges to zero, the rebalancing dates concentrate near the maturity. To visualize the impact of parameter b on the rebalancing dates grid, we have reported on Figure 3.2 the sequences of rebalancing dates generated by π_N^b for different values of b.

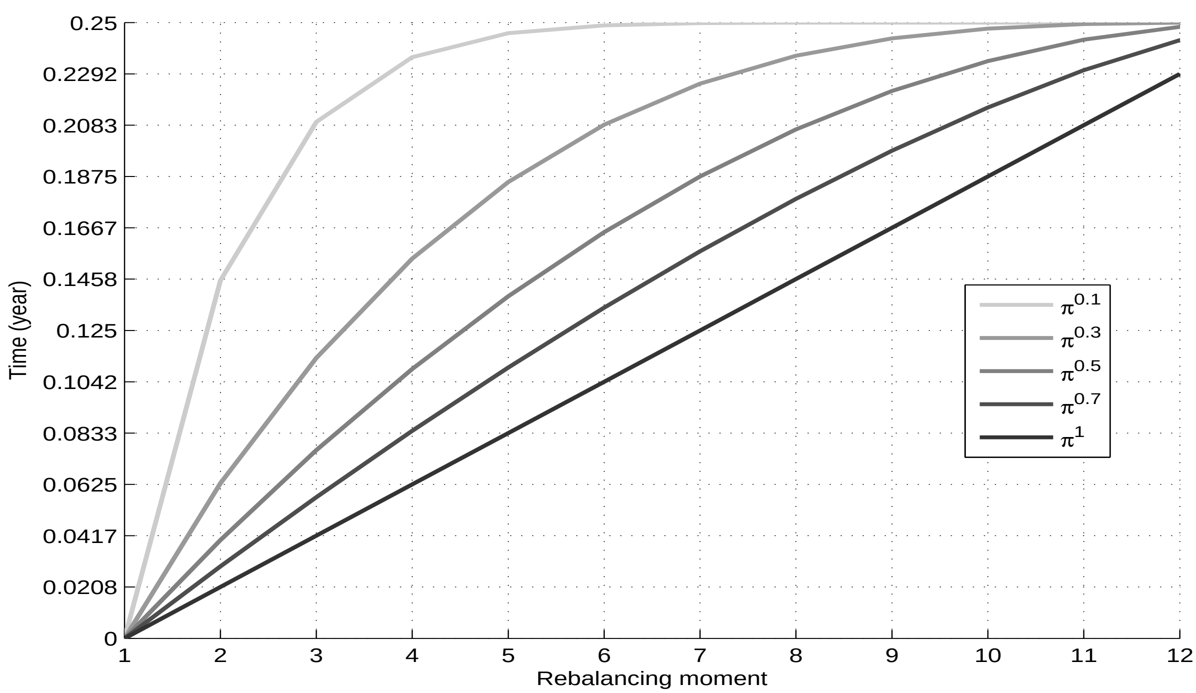


Figure 3.2: Sequences of rebalancing dates for different values of b, for $N = 12$.

We have reported on Figure 3.3 the standard deviation of the Variance Optimal hedging error for different values of coefficient C and different choices of rebalancing grids. More precisely, we have considered three types of rebalancing grids, for $N = 12$ rebalancing dates.

1. Equidistant rebalancing dates (corresponding to $\pi^{1,N}$);
2. $\pi^{b^*,N}$ where b^* is obtained by minimizing the Variance Optimal hedging error w.r.t. to parameter b;
3. The non parametric optimal grid π^* obtained by minimizing the Variance Optimal hedging error w.r.t. the N rebalancing dates.

Notice that in both cases the optimal (parametric and non parametric) grid is estimated by an optimization algorithm based on Newton's method.
First, one can notice that for any choice of rebalancing grid, the hedging error increases when C decreases. Hence, one can conclude, as expected, that the *degree of incompleteness* increases when the tails of log-returns distribution get heavier.
Besides, one can notice that the parametrization (5.39) of the rebalancing grid seems remarkably relevant since the optimal parametric grid π^{b^*} achieves similar performances as the optimal non-parametric grid π^*.
Moreover, we observe that the hedging error can be noticeably reduced by optimizing the rebalancing dates essentially for $C \geq 1$ i.e. around the Gaussian case. In these cases, one can observe on Figure 3.4 that the optimal rebalancing grid is noticeably different from the uniform grid since rebalancing dates are much more concentrated near maturity. This confirms the result of [39] that shows that, in the Gaussian case, taking a non uniform rebalancing grid (corresponding to $b = 0.5$) allows to obtain a hedging error with the convergence order for the L^2 norm of $N^{-1/2}$ (up to a log factor) improving the rate $N^{-1/4}$ achieved with a uniform rebalancing grid (i.e. $b = 1$), obtained in [43]. However, it is interesting to notice that this phenomenon is less pronounced when the tails of the log-returns distribution get heavier. In particular, one can observe on Figure 3.5 that the hedging error gets less sensitive to the rebalancing grid when C decreases even if the optimal grid seems to get closer to the uniform grid.

	$C = 2$	$C = 1$	$C = 0.2$	$C = 0.14$
$10 \times STD_{VO(\pi^*)}$	1.483 (30.82)	1.652 (34.33)	2.663 (54.80)	3.017 (61.53)
$10 \times STD_{VO(\pi^{b*})}$	1.520 (31.58)	1.685 (35.01)	2.665 (54.84)	3.017 (61.53)
$10 \times STD_{VO(\pi^1)}$	1.892 (39.32)	1.952 (40.56)	2.691 (55.38)	3.028 (61.76)
$V_0(\pi^1)$	0.4903	0.4859	0.4813	0.4812
$V_0(\pi^*)$	0.4903	0.4860	0.4814	0.4813
b^*	0.4078	0.4394	0.6106	0.6710

Figure 3.3: Standard deviation of the Variance Optimal hedging error ($\times 10$) (reported within parenthesis in percent of the option value $V_0(\pi^1)$), initial capitals, optimal grid parameters, for different choices of parameters C and b with $N = 12$ and $K = 99$ (Digital option).

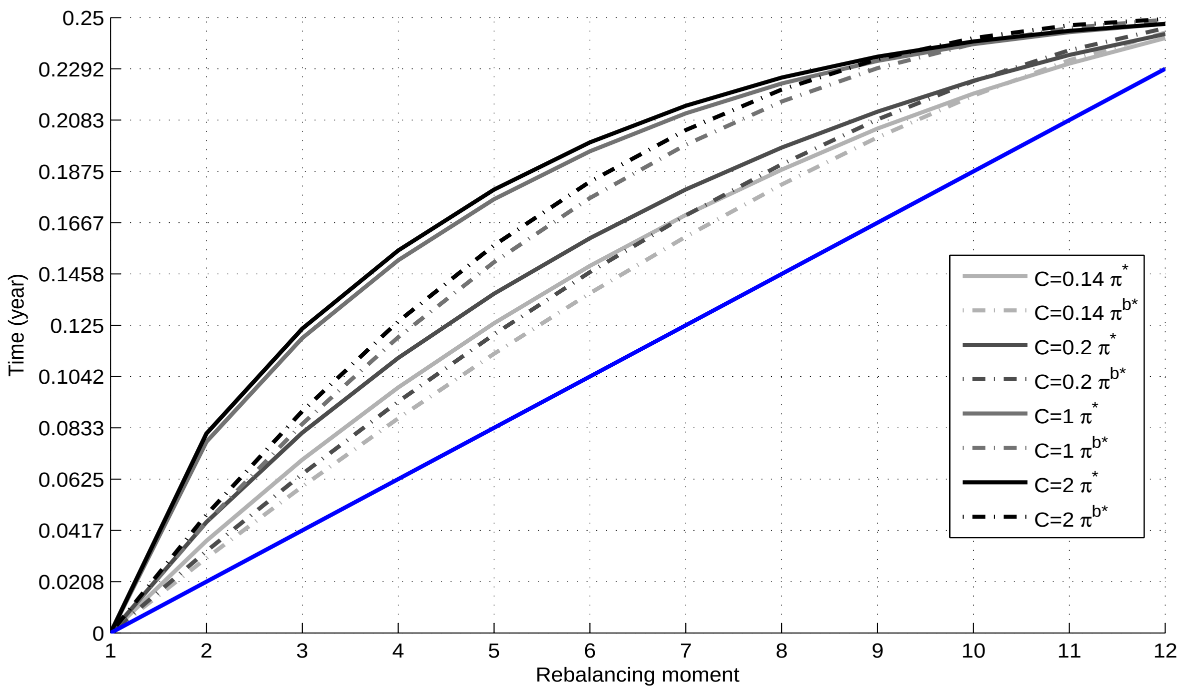


Figure 3.4: Parametric and non parametric optimal rebalancing grids for different choices of parameter C with $N = 12$ and $K = 99$ (Digital option).

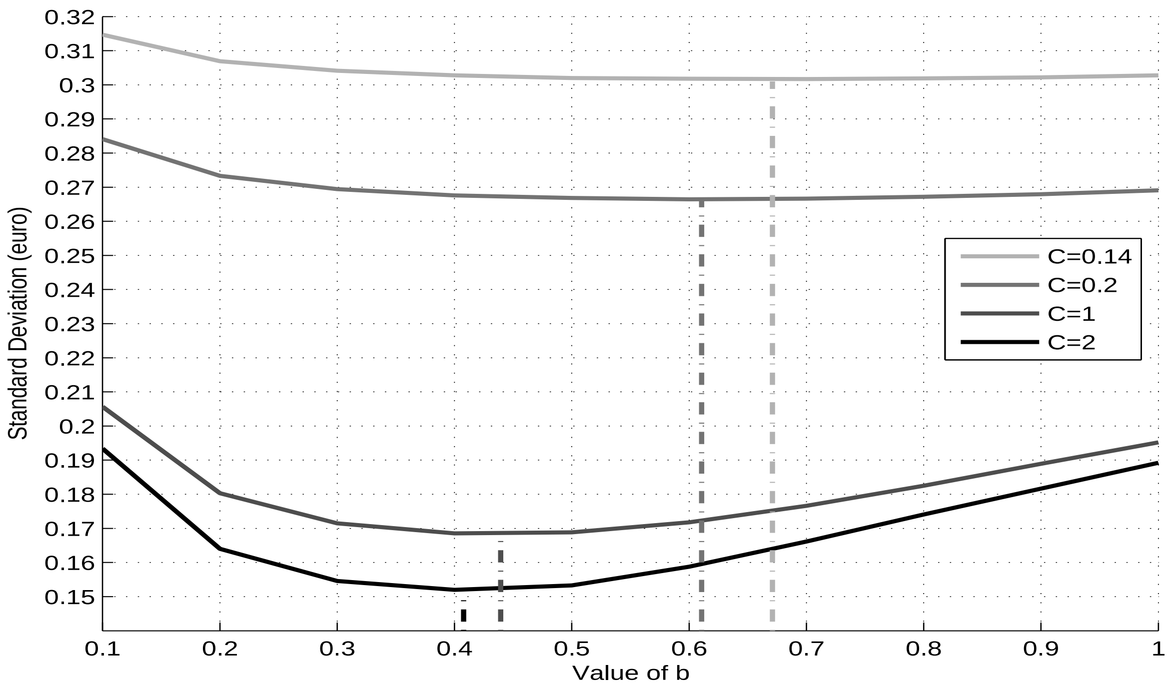


Figure 3.5: Standard deviation of the Variance Optimal hedging error as a function of b, for different choices of parameter C (b^* being indicated by the dashed line abscissa) with $N = 12$ and $K = 99$ (Digital option).

3.5.2 The case of electricity forward prices

We consider the problem of hedging and pricing a European call, with payoff $(F_T^{T_d} - K)_+$, on an electricity forward, with a maturity $T = 0.25$ of three month. The maturity T is supposed to be equal to the delivery date of the forward contract $T = T_d$. Because of non-storability of electricity, the hedging instrument is the corresponding forward contract. Then we set $S_t^c = F_t^T$, where the forward price F^T is supposed to follow the NIG one factor model (1.1) with $m \equiv 0, \sigma_L = 0$ and $\sigma_s = \sigma > 0$. This gives

$$S_t^c = e^{X_t^c}\,, \quad \text{where } X_t^c = \int_0^t \sigma e^{-\lambda(T-u)} d\Lambda_u \quad \text{where } \Lambda \text{ is a NIG process with} \quad \Lambda_1 \sim \text{NIG}(\alpha, \beta, \delta, \mu)\,. \tag{5.40}$$

Given $N+1$ discrete dates $0 = t_0 < t_1 < \cdots < t_N = T$, we consider the discrete process $X = X^N$ where $X_k = X_{t_k}^c, 0 \leq k \leq N$. We denote again by $z \mapsto m(z,k)$ the cumulant generating function associated with the increment $\Delta X_k = X_k - X_{k-1}$ for $k \in \{1, \cdots N\}$. That function and its domain can be deduced from Lemma 2.3.24 and Proposition 2.6.2 in chapter 2 of this thesis, see also(5.45). The domain D contains $\tilde{D} := [-\frac{\alpha+\beta}{\sigma}, \frac{\alpha-\beta}{\sigma}] + i\mathbb{R}$ and given for any $z \in \tilde{D}, k = 0, \ldots, N$,

$$\begin{aligned} m(z,k) &= \mathbb{E}[\exp(z \int_{t_{k-1}}^{t_k} \sigma e^{-\lambda(T-u)} d\Lambda_u)] \\ &= \exp\left(\int_{t_{k-1}}^{t_k} \kappa^{\Lambda}(z\sigma e^{-\lambda(T-u)}) du \right)\,, \quad \text{with } z_u = z\sigma e^{-\lambda(T-u)} \\ &= \exp\left(\int_{t_{k-1}}^{t_k} \left[\mu z_u + \delta\left(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + z_u)^2}\right)\right] du \right)\,. \end{aligned} \tag{5.41}$$

Hence Assumption 13 1. is obviously satisfied since $\lambda \neq 0$ and Assumption 13 2. is verified as soon as $\sigma \leq \frac{\alpha-\beta}{2}$; thanks to Remark 3.3.15, Assumption 14 is automatically verified for the call representation given by Lemma 2.4.26.

Parameters are estimated on the same data as in the previous section, with *Month-ahead base* forward prices of the French Power market in 2007. For the distribution of Λ_1 this yields the following parameters

$$\alpha = 15.81\,,\ \beta = -1.581\,,\ \delta = 15.57\,,\ \mu = 1.56\ ,$$

corresponding to a standard and centered NIG distribution with a skewness of -0.019 and excess kurtosis 0.013. The estimated annual short-term volatility and mean-reverting rate are $\sigma = 57.47\%$ and $\lambda = 3$.

We have reported on Figure 3.6, the standard deviation of the hedging error as a function of the number of rebalancing dates for four types of hedging strategies.

- **Variance Optimal** strategy (VO) with the **uniform rebalancing grid (dark line)** and with the **optimal rebalancing grid** π^* **(dark dashed line)**. Both variances are computed using formula (4.20) applied to the process (5.40);
- **Black-Scholes** strategy (BS) implemented at the discrete instants of the **uniform rebalancing grid (light line)** and of the **rebalancing grid** π^* **(optimal for the Variance Optimal strategy) (light dashed line)**. Both variances are computed by a Monte Carlo approximation using 10^5 independent simulations of the process (5.40).

Notice that simulations of model (5.40) (resp. computations of X's cumulant generating function) is performed using a stochastic (resp. deterministic) Euler scheme with 100 discretization steps of the interval $[0, T]$.
Observing Figure 3.6, one can notice that, as expected, in all cases, the hedging error decreases when the number of trading dates increases. Observing the continuous lines, corresponding to a uniform rebalancing grid, one can notice the remarkable robustness of the Black-Scholes strategy. Indeed, in spite of the non Gaussianity of log-returns and the discreteness of the rebalancing grid, the Black-Scholes strategy is still quasi optimal in terms of variance.
Besides, in this case, the impact of the choice of the rebalancing grid seems to be more important than the choice of log-returns distribution (Gaussian or Normal Inverse Gaussian). For instance, using the VO strategy with the optimal rebalancing grid π^* instead of π^1 allows to reduce 9% (for $N = 10$) of the hedging error standard deviation.
However, contrarily to what we observe with the uniform grid π^1, the BS strategy shows performances that differ noticeably from the VO performances, when implemented at the rebalancing times π^*. This suggests that the *BS optimal rebalancing grid* (in terms of variance) is probably noticeably different from π^*. It would be interesting to minimize the BS hedging error w.r.t. the rebalancing grid and verify if it achieves similar performances as $VO(\pi^*)$. But this requires a great amount of computing time since the standard deviation of the BS hedging error is approximated by Monte Carlo simulations. An alternative would be to extend results of [5] to non-stationary log-returns, to derive a quasi-explicit formula for the variance of the BS hedging error. Indeed, in [5], the authors uses the Laplace transform approach, to derive quasi-explicit formulae for the mean squared hedging error of various dis-

crete time hedging strategies including Black-Scholes delta when applied to Lévy log-returns models. Finally, one can observe on Figure 3.7 that here again, the parametrization (5.39) of the rebalancing grid seems to be particularly well suited since it achieves minimal hedging errors comparable to the one achieved with the nonparametric optimal grid π^*.

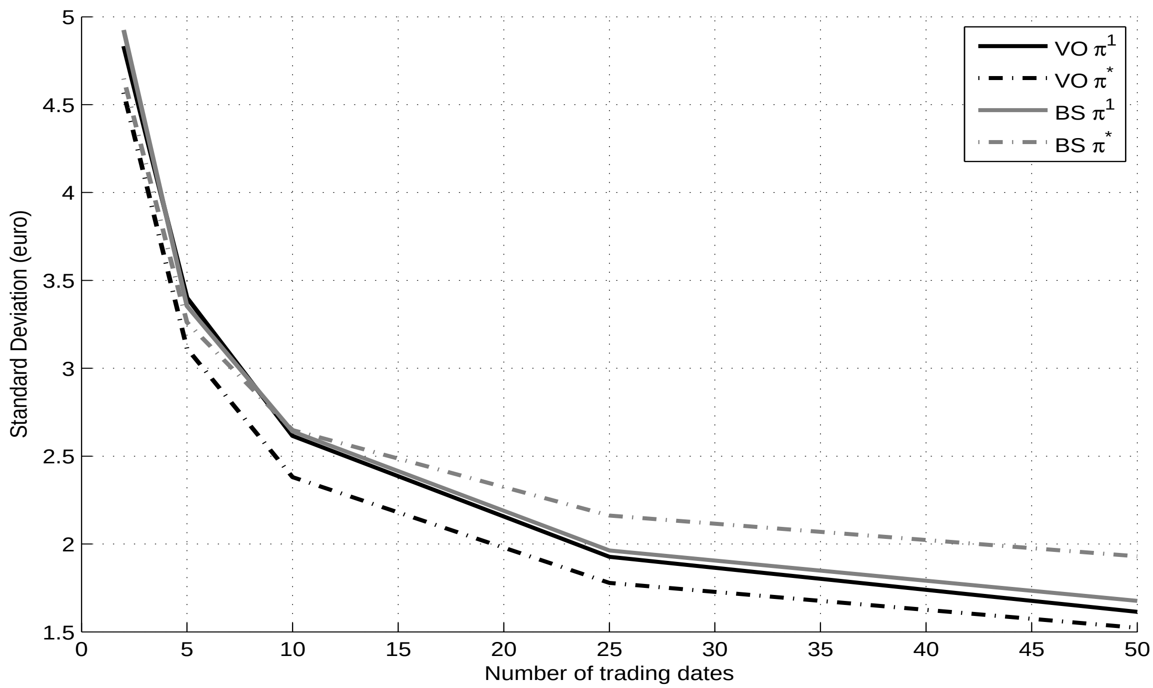


Figure 3.6: Standard deviation of the hedging errors as a function of the number of rebalancing dates N, for $K = 99$ (Call option).

To analyse the impact of the rate of volatility increase on the optimal rebalancing grid, we have computed the hedging error standard deviation for several values of parameter λ chosing the corresponding volatility parameter σ such that $Var(X_T) = \frac{\sigma^2}{2\lambda}(1-e^{-2\lambda T})$ is fixed. The resulting pairs (λ, σ) are reported on Figure 3.8. Coupling those parameters allows us to obtain comparable options for different parameters λ; at least this ensures a fixed initial capital in the BS framework (with $V0_{BS}(\pi^1) = 8.7037$).

On Figure 3.9, we have reported the optimal grid parameter b^* minimizing the standard deviation of the VO hedging error for different values of λ. As expected, when λ increases, i.e. when the volatility increases more rapidly near the maturity, then b^* decreases indicating that

	$N = 2$	$N = 5$	$N = 10$	$N = 25$	$N = 50$
$STD_{VO(\pi^*)}$	4.5683 (53.23)	3.1129 (36.10)	2.3807 (27.56)	1.7790 (20.57)	1.5233 (17.61)
$STD_{VO(\pi^{b*})}$	4.57167 (53.27)	3.1550 (36.59)	2.4186 (28.00)	1.8023 (20.84)	1.5354 (17.75)
$STD_{VO(\pi^1)}$	4.8331 (56.32)	3.4012 (39.44)	2.6154 (30.28)	1.9275 (22.29)	1.6145 (18.66)
$STD_{BS(\pi^1)}$	4.9252 (57.39)	3.3536 (38.89)	2.6405 (30.57)	1.9631 (22.70)	1.6769 (19.39)
$STD_{BS(\pi^*)}$	4.6486 (54.17)	3.2611 (37.82)	2.6478 (30.65)	2.1619 (25.00)	1.9303 (22.32)
$V_0(\pi^1)$	8.5818	8.6232	8.6380	8.6469	8.6499
$V_0(\pi^*)$	8.5895	8.6275	8.6406	8.6493	8.6531
b^*	0.5917	0.6298	0.6284	0.6203	0.6172

Figure 3.7: Standard deviation of the Variance Optimal hedging error (reported within parenthesis in percent of the option value $V_0(\pi^1)$), initial capitals, optimal grid parameters, for different choices of rebalancing dates N (Call option).

the optimal rebalancing dates concentrate near the maturity. On Figure 3.10, one can observe that the hedging error increases with λ even when the rebalancing dates are optimized. However, optimizing the rebalancing dates allows to reduce noticeably the hedging error, specifically for high values of λ. For instance, it allows to reduce 7.5% of the error standard deviation when $\lambda = 3$ and 17.9% when $\lambda = 9$.

λ	1	2	3	6	9
σ	0.4662	0.5202	0.5747	0.7349	0.8823
$V_0(\pi^1)$	8.6630	8.6511	8.6380	8.5936	8.5450
$V_0(\pi^{b*})$	8.6615	8.6516	8.6406	8.6022	8.5597

Figure 3.8: Short term volatility σ (s.t. $Var(X_T) = \frac{\sigma^2}{2\lambda}(1 - e^{-2\lambda T})$ is fixed) and initial capitals for different values of parameter λ with $N = 10$ and $K = 99$ (Call option).

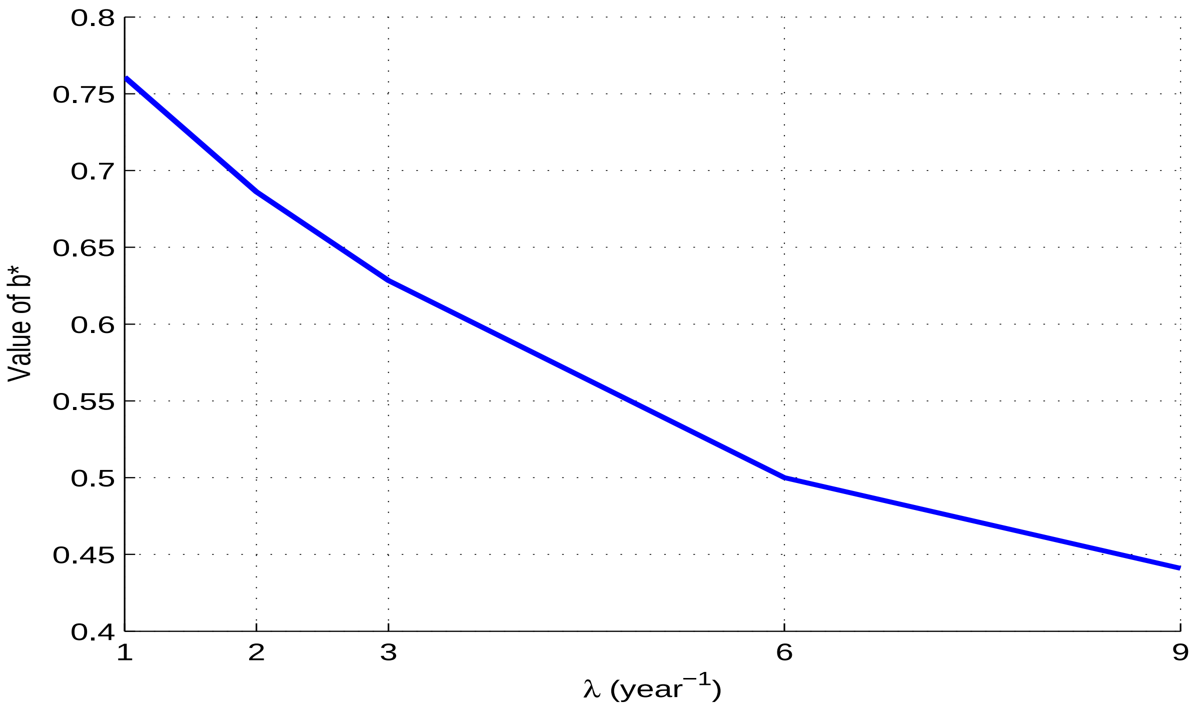


Figure 3.9: Optimal rebalancing grid parameter b^* as a function of λ for $K = 99$ and $N = 10$ (Call option).

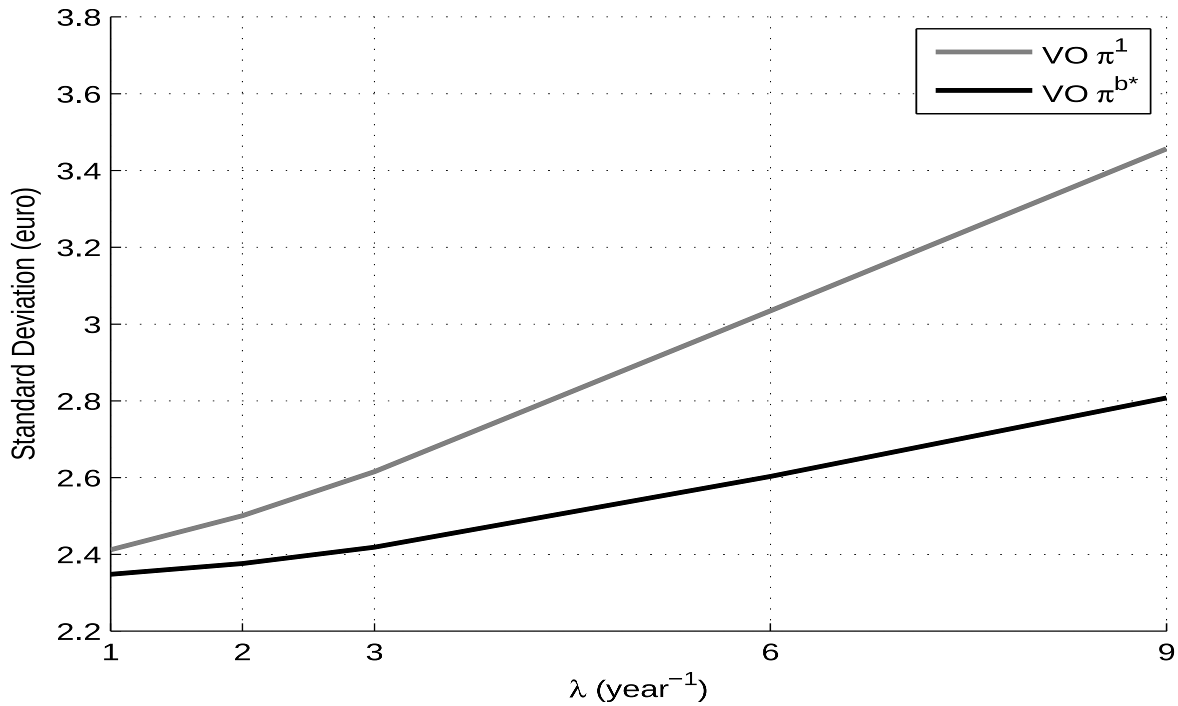


Figure 3.10: Standard deviation of the hedging error as a function of λ for $K = 99$ and $N = 10$ (Call option).

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Appendix: The Normal Inverse Gaussian distribution

The Normal Inverse Gaussian (NIG) distribution is a specific subclass of the Generalized Hyperbolic family introduced by Bandorf–Nielsen in 1977, see for instance [6]. The density of a Normal Inverse Gaussian distribution of parameters $(\alpha, \beta, \delta, \mu)$ is given by

$$f_{NIG}(x; \alpha, \beta, \delta, \mu) = \frac{\alpha}{\pi} \exp\left(\delta\sqrt{\alpha^2 - \beta^2} + \beta(x - \mu)\right) \frac{K_1\left(\alpha\delta\sqrt{1 + (x-\mu)^2/\delta^2}\right)}{\sqrt{1 + (x-\mu)^2/\delta^2}} , \quad \text{for any } x \in \mathbb{R} , \tag{5.42}$$

where K_1 denotes the Bessel function of the third type with index 1 and where the parameters are such that $\delta > 0$, $\alpha > 0$ and $\alpha > |\beta|$. Afterwards, $\mathrm{NIG}(\alpha, \beta, \delta, \mu)$ will denote the Normal Inverse Gaussian distribution of parameters $(\alpha, \beta, \delta, \mu)$.

A useful property of the NIG distribution is its stability under convolution i.e.

$$NIG(\alpha, \beta, \delta_1, \mu_1) * NIG(\alpha, \beta, \delta_2, \mu_2) = NIG(\alpha, \beta, \delta_1 + \delta_2, \mu_1 + \mu_2) .$$

This property shared with the Gaussian distribution allows to simplifies many computations.

If X is a $\mathrm{NIG}(\alpha, \beta, \delta, \mu)$ random variable then for any $a \in \mathbb{R}^+$ and $b \in \mathbb{R}$, $Y = aX + b$ is also a NIG random variable with parameters $(\alpha/a, \beta/a, a\delta, a\mu + b)$.

The mean and the variance associated to a $\mathrm{NIG}(\alpha, \beta, \delta, \mu)$ random variable X are given by,

$$\mathbb{E}X = \mu + \frac{\delta\beta}{\gamma} , \quad \mathrm{Var}X = \frac{\delta\alpha^2}{\gamma^3} , \quad \text{with} \quad \gamma = \sqrt{\alpha^2 - \beta^2} . \tag{5.43}$$

The characteristic function of the NIG distribution is given by $\exp(\Psi_{NIG})$ where Ψ_{NIG} verifies

$$\Psi_{NIG}(u) = \log \mathbb{E}\big[\exp(iuX)\big] = i\mu u + \delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + iu)^2}) \quad \text{for any} \quad u \in \mathbb{R} . \tag{5.44}$$

The moment generating function of the NIG distribution is particularly simple,

$$\kappa_{NIG}(z) = \log \mathbb{E}[\exp(zX)] = \mu z + \delta\left(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + z)^2}\right) , \quad \text{for } Re(z) \in [-(\alpha+\beta); \alpha-\beta] . \tag{5.45}$$

The Lévy measure of the NIG distribution is given by

$$F_{NIG}(dx) = e^{\beta x} \frac{\delta\alpha}{\pi|x|} K_1(\alpha|x|)\, dx \quad \text{for any } x \in \mathbb{R} . \tag{5.46}$$

Notice that the Lévy measure does not depend on parameter μ.

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RESUME en français: La thèse porte sur une décomposition explicite de Föllmer-Schweizer d'une classe importante d'actifs conditionnels lorsque le cours du sous-jacent est un processus à accroissements indépendants ou une exponentielle de tels processus. Ceci permet de mettre en oeuvre un algorithme efficace pour établir des stratégies optimales dans le cadre de la couverture quadratique. Ces résultats ont été implémentés dans le cas du marché de l'électricité.

TITRE en italien: Copertura sulla base dello scarto quadratico medio nei mercati incompleti per dei processi a incrementi indipendenti e applicazioni al mercato elettrico.

RESUME en italien: In questa tesi di dottorato di ricerca vengono calcolate esplicitamente le scomposizioni dette di *Föllmer-Schweizer* per una famiglia significativa di opzioni finanziarie quando il prezzo del soggiacente é un processo a incrementi indipendenti o un esponenziale di tali processi. Le formule ottenute permettono di produrre un algoritmo efficiente per la risoluzione del problema della copertura che minimizza lo scarto quadratico medio nei mercati incompleti. I risultati sono stati implementati numericamente nell'ambito del mercato elettrico.

TITRE en anglais: Variance Optimal Hedging in incomplete market for processes with independent increments and applications to electricity market.

RESUME en anglais: For a large class of vanilla contingent claims, we establish an explicit Föllmer-Schweizer decomposition when the underlying is a process with independent increments (PII) and an exponential of a PII process. This allows to provide an efficient algorithm for solving the mean variance hedging problem. Applications to models derived from the electricity market are performed.

DISCIPLINE: Mathématiques (Université Paris 13), Metodi matematici per l'economia, la finanza e l'impresa (LUISS GUIDO CARLI).

MOTS-CLES: Variance-optimal hedging, Föllmer-Schweizer decomposition, Lévy process, Cumulative generating function, Characteristic function, Normal Inverse Gaussian process, Electricity markets, Incomplete Markets, Process with independent increments, trading dates optimization.

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